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## **“Adaptive H-Infinity Control for Nonlinear Systems: A Dissipation Theoretical Approach”**

Wei-Min Lu and Andy Packard

**Control and Dynamical Systems**  
California Institute of Technology  
Pasadena, California 91125

# Adaptive $\mathcal{H}_\infty$ -Control for Nonlinear Systems: A Dissipation Theoretical Approach

Wei-Min Lu<sup>\*</sup> and Andy Packard<sup>†</sup>

## Abstract

The adaptive  $\mathcal{H}_\infty$ -control problem for parameter-dependent nonlinear systems with full information feedback is considered. The techniques from dissipation theory as well as the vector and parameter projection methods are used to derive the adaptive  $\mathcal{H}_\infty$ -control laws. Both of the projection techniques are rigorously treated. The adaptive robust stabilization for nonlinear systems with  $\mathcal{L}_2$ -gain bounded uncertainties is investigated.

## 1 Introduction

The  $\mathcal{H}_\infty$ -control problem for dynamical systems with external disturbances is to design feedback controllers which make the resulting systems to have small  $\mathcal{L}_2$ -gains (or  $\mathcal{H}_\infty$ -norms for linear systems) such that the external disturbances are attenuated [9, 8, 39, 4, 36, 3, 13, 18]. In this paper, the  $\mathcal{H}_\infty$ -control problem for nonlinear systems which additionally depend on unknown parameters is considered by the use of adaptive control schemes with full information feedback.

It is known from dissipation theory that a dynamical system has bounded  $\mathcal{L}_2$ -gain if the system is dissipative in some sense (which will be made precise in the body) [37]. Therefore, essentially, the  $\mathcal{H}_\infty$ -control design requires designing controller such that the resulting closed loop system is dissipative, and correspondingly there exists a storage function [18, 3]. In this paper, we will deal with the adaptive  $\mathcal{H}_\infty$ -control problem using dissipation theoretical techniques. Our starting point for solving adaptive  $\mathcal{H}_\infty$ -control problem is that the corresponding deterministic problem, where the parameters are known, has parameterized solutions. Therefore, if the parameter is known, then the existence of  $\mathcal{H}_\infty$ -control solutions is equivalent to the existence of parameterized feedback controllers such that the resulting parameterized system is dissipative, which is characterized in terms of the solutions of parameter-dependent Hamilton-Jacobi inequalities. This kind of control scheme has a gain-scheduling interpretation [24, 20, 6, 1]. However, if the parameters are unknown, then the parameterized controller can no longer be used. We instead need to design a parameter estimation mechanism to adjust the parameters for implementation of the parameterized controller (possibly modified). To achieve adaptive  $\mathcal{H}_\infty$ -performance, both the parameterized controller and the parameter-adjustment mechanism are designed properly such that the augmented system, whose states include the states of plant and parameter-adjustment mechanism, is dissipative; correspondingly, there exists a storage function for the resulting adaptive  $\mathcal{H}_\infty$ -control system. In fact, in

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<sup>\*</sup>Berkeley Center for Control and Identification, University of California, Berkeley, CA 94720 and Electrical Engineering 116-81, Caltech, Pasadena, CA 91125

<sup>†</sup>Berkeley Center for Control and Identification and Department of Mechanical Engineering, University of California, Berkeley, CA 94720

this paper, the storage function is explicitly constructed by the use of similar techniques in the Lyapunov design of adaptive controller [26, 15, 29, 38, 17]; both the parameterized controller and the parameter-adjustment mechanism are then naturally constructed. This constructive approach gives sufficient conditions for the adaptive  $\mathcal{H}_\infty$ -control problem to have solutions. The similar approach is also used to address the adaptive robust control problem for a nonlinear uncertain systems with  $\mathcal{L}_2$ -gain bounded dynamical uncertainty, where the objective is to design an adaptive robust controller such that the resulting uncertain system is robustly asymptotically stable.

In applications, the unknown parameters of the parameterized plants are usually bounded. In this paper, it is assumed that allowable parameter sets are compact and convex, but not necessarily smooth, for instance, a cube in the parameter space. During the adaptation, one of the requirements is that the parameter-adjustment mechanism of an adaptive control system keep the adjusted parameters in the parameter sets so as not to invalidate the solvability conditions. In this paper, both vector and direct parameter projection techniques are used to achieve this goal. The vector projection, which was originally introduced as a gradient projection method to generate the feasible directions in constrained optimization [32, 21], is probably the most extensively used projection technique in adaptive parameter estimation and adaptive control [10, 29, 28, 22, 11, 17]. However, in these cases, the projections are considered only for smooth sets. In this paper, we will generalize the vector projection to a more general setting such that the non-smooth parameter sets are allowed. The direct parameter projection is relatively new in adaptive control (another version appeared in [7]). It will be seen that this technique is suitable for the adaptive control problems where integral performance specifications are involved, in particular adaptive  $\mathcal{H}_\infty$ -control problem. The two projections not only play a very important role in adaptive control problems, but also are of interest in their own right; in this paper, they are treated in detail using techniques from non-smooth analysis and viability theory [2, 31].

Other work related to the adaptive  $\mathcal{H}_\infty$ -control includes, for example, parameter-estimation based approaches by Krause *et al.* [16] and Basar *et al.* [5], a game-theoretical approach by Didinsky-Basar [7], a Lyapunov design approach by Yang *et al.* [38], a back-stepping approach by Pan-Basar [25], and a problem with finite parameter set by Rangan-Poolla [30]. In this paper, the emphasis is the use of both dissipation and projection techniques; the existence of adaptive  $\mathcal{H}_\infty$ -controllers are characterized in terms of solutions of parameter-dependent Hamilton-Jacobi inequalities, and the adaptive controllers are constructed. The organization of this paper is as follows. In Section 2, the parameterized  $\mathcal{H}_\infty$ -control design for parameterized systems is considered. The material serves as a review of deterministic  $\mathcal{H}_\infty$ -control results where the emphasis is on its connection with dissipation theory; it is also preliminary for later discussions. In Section 3, both the vector and the direct parameter projections techniques are rigorously treated with respect to compact, convex, but possibly nonsmooth parameter sets. In Section 4, an adaptive  $\mathcal{H}_\infty$ -control problem is stated. In Section 5, the solutions for the adaptive  $\mathcal{H}_\infty$ -control problem are derived for two cases, i.e., when the original storage functions are independent of and dependent on the parameters respectively. Both vector and direct parameter projection techniques are used in the derivation of adaptive control laws. The stability and asymptotic property for the adaptive  $\mathcal{H}_\infty$ -control systems are discussed. In Section 6, the adaptive robust stabilization problem for a nonlinear uncertain system with  $\mathcal{L}_2$ -gain bounded dynamical uncertainty is addressed. In Section 7, a simple illustrative example is provided, and some comments follow.

## 2 Parameterized $\mathcal{H}_\infty$ -Control Design

In this section, we will consider the  $\mathcal{H}_\infty$ -control design of parameterized systems when the parameters are constant and known. We will design a state-feedback control scheme where the control gain also depends on the parameters. This parameterized  $\mathcal{H}_\infty$ -control scheme gives a gain scheduling interpretation; the implementation of such controllers requires that the parameters be available to the control during system operation. The material in this section also serves as a review of the  $\mathcal{H}_\infty$ -control results for deterministic systems. We will emphasize its connection with the dissipation theory [37].

### 2.1 $\mathcal{L}_2$ -Gains and Dissipativity

In this subsection, we will review some results about the  $\mathcal{L}_2$ -gain analysis of a dynamical system; in particular, its relation to the dissipativity of the system is investigated. This issue is also discussed in [36, 18].

Consider the following affine nonlinear time-invariant system:

$$\begin{cases} \dot{x} = f(x) + g(x)w \\ z = h(x) + k(x)w \end{cases} \quad (1)$$

where  $x \in \mathbf{R}^n$  is the state vector,  $w \in \mathbf{R}^p$  and  $z \in \mathbf{R}^q$  are the input and the output vectors, respectively. We will assume  $f, g, h, k \in \mathbf{C}^2$ , and  $f(0) = 0, h(0) = 0$ . Therefore,  $0 \in \mathbf{R}^n$  is the equilibrium of the system with  $w = 0$ . The state transition function  $\phi : \mathbf{R}^+ \times \mathbf{R}^n \times \mathcal{L}_2^p(\mathbf{R}^+) \rightarrow \mathbf{R}^n$  is so defined that  $x = \phi(T, x_0, w^*)$  means that the given system evolves from initial state  $x_0$  to state  $x$  in time  $T$  under the input  $w^* \in \mathcal{L}_2^p(\mathbf{R}^+)$ . A system is **reachable** from 0 if for all  $x \in \mathbf{X}$ , there exist  $T \in \mathbf{R}^+$  and  $w^*(t) \in \mathcal{L}_2[0, T]$  such that  $x = \phi(T, 0, w^*)$ . The following definition of nonlinear  $\mathcal{L}_2$ -gains is standard [36]:

**Definition 2.1** *The given system (1) is said to have  $\mathcal{L}_2$ -gain less than or equal to  $\gamma$  if*

$$\int_0^T \|z(t)\|^2 dt \leq \gamma^2 \int_0^T \|w(t)\|^2 dt$$

for all  $T \geq 0$  and  $w(t) \in \mathcal{L}_2[0, T]$ .

Note that in the above definition, we take the initial state  $x(0) = 0$ . In the following, we will consider a system with  $\mathcal{L}_2$ -gain bound normalized as 1. It is known that a system having  $\mathcal{L}_2$ -gain less than or equal to 1 is dissipative with respect to the supply rate  $\|w(t)\|^2 - \|z(t)\|^2$  [37]. Therefore, the system has  $\mathcal{L}_2$ -gain less than or equal to 1 if and only if there exists a non-negative function  $V : \mathbf{R}^n \rightarrow \mathbf{R}^+$  such that the following dissipation inequality hold,

$$V(x(T)) - V(x(0)) \leq \int_0^T (\|w(t)\|^2 - \|z(t)\|^2) dt; \quad (2)$$

any function  $V$  is called a **storage function** [37]. Define a function  $V_a : \mathbf{R}^n \rightarrow \mathbf{R}$

$$V_a(x) := \sup_{w \in \mathcal{L}_2[0, \tau], \tau \geq 0, x(0)=x} - \int_0^\tau (\|z(t)\|^2 - \|w(t)\|^2) dt; \quad (3)$$

note that  $V_a(x) \geq 0$  for all  $x \in \mathbf{X}$ . Moreover, we have the following Willems' Lemma [37, 18].

**Lemma 2.2 (Willems' Lemma)** Consider system (1); suppose it is reachable from 0. Then it has  $\mathcal{L}_2$ -gain  $\leq 1$  if and only if  $V_a(x) < \infty$ , for all  $x \in \mathbf{X}$ . Moreover,  $V_a(x)$  is a storage function, and for all storage function  $V$  satisfying (2),  $V(x) \geq V_a(x)$  for all  $x \in \mathbf{R}^n$ .

Next, we consider the nonlinear system (1) with the input-affine structure. Given a  $\mathbf{C}^1$  function  $V : \mathbf{R}^n \rightarrow \mathbf{R}$ , suppose  $R(x) := I - k^T(x)k(x) > 0$  for all  $x \in \mathbf{R}^n$ ; define

$$\begin{aligned} \mathcal{H}(V, x) := & \frac{\partial V}{\partial x}(x)(f(x) + g(x)R^{-1}(x)k^T(x)h(x)) + \\ & + \frac{1}{4} \frac{\partial V}{\partial x}(x)g(x)R^{-1}(x)g^T(x) \frac{\partial V^T}{\partial x}(x) + h^T(x)(I - k(x)k^T(x))^{-1}h(x) \end{aligned} \quad (4)$$

Suppose the system (1) has  $\mathcal{L}_2$ -gain  $\leq 1$ . If a storage function is differentiable, then it satisfies the Hamilton-Jacobi inequality:  $\mathcal{H}(V, x) \leq 0$ . In particular, if the storage function defined by (3) is differentiable, then the equality holds, i.e.,  $\mathcal{H}(V_a, x) = 0$ .

The following proposition characterizes sufficient conditions for a nonlinear system with input-affine structure to have  $\mathcal{L}_2$ -gain  $\leq 1$ .

**Proposition 2.3** Consider the given system (1) with  $\mathcal{H}(V, x)$  defined as (4). Then

$$\dot{V}(x) = \|w(t)\|^2 - \|z(t)\|^2 - \left\| R^{1/2}(x)(w - w^*(x)) \right\|^2 + \mathcal{H}(V, x),$$

where  $w^*(x) := R^{-1}(x)(k^T(x)h(x) + \frac{1}{2}g^T(x)\frac{\partial V^T}{\partial x}(x))$ . In particular, if  $V(x) \geq 0$  with  $V(0) = 0$  satisfies Hamilton-Jacobi inequality:  $\mathcal{H}(V, x) \leq 0$ , then  $G$  has  $\mathcal{L}_2$ -gain  $\leq 1$ .

## 2.2 Parameterized State-Feedback $\mathcal{H}_\infty$ -Control

Parameterized  $\mathcal{H}_\infty$ -controllers are parameter-dependent, and have a gain scheduling interpretation. The standard block diagram for such control system is illustrated in Figure 1, where  $G(\theta)$  is the parameterized plant and  $K(\theta)$  is the parameterized controller; both  $G(\theta)$  and  $K(\theta)$  are dependent on the same parameter  $\theta$ ;  $w$  is the vector of exogenous disturbance inputs and  $u$  is the vector of control inputs;  $z$  is the the vector of outputs to be regulated; and  $y$  is the vector of measured outputs based on which the control action is generated.

In this section,  $G(\theta)$  has the following state-space realization where the disturbance  $w$  enters the system in an affine form:

$$G(\theta) : \begin{cases} \dot{x} &= f(x, u, \theta) + g(x, \theta)w \\ z &= h(x, u, \theta) + k(x, \theta)w \\ y &= x \end{cases} \quad (5)$$

where  $f, g, h, k \in \mathbf{C}^0$ ,  $\theta \in \Theta \subset \mathbf{R}^r$  is the scheduled parameter which is assumed to be constant, and  $f(0, 0, \theta) = 0, h(0, 0, \theta) = 0$  for all  $\theta \in \Theta$ ;  $x$ ,  $w$ ,  $u$ ,  $z$ , and  $y$  are state, disturbance, control input, regulated output, and measured output vectors with dimensions  $n$ ,  $p_1$ ,  $p_2$ ,  $q_1$ , and  $q_2$ , respectively. In the gain scheduling design in this section, we assume the controller has access to the state  $x$  and the parameter  $\theta$ . The admissible state feedback set is defined as

$$\mathcal{K} = \{F : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^{p_2} \mid F \in \mathbf{C}^0, F(0, \theta) = 0, \forall \theta \in \Theta\} \quad (6)$$

We have the following control problem.

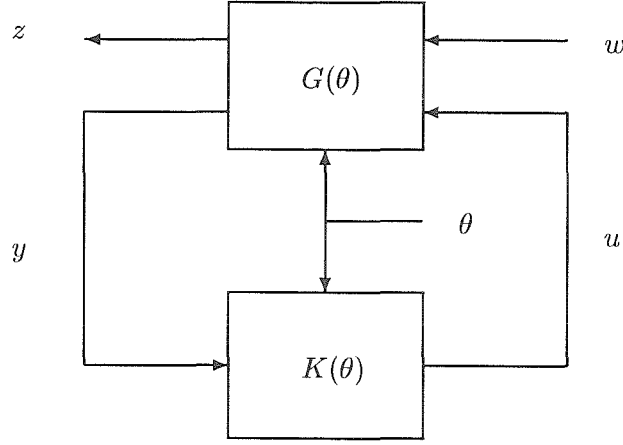


Figure 1: Gain-Scheduled  $\mathcal{H}_\infty$ -Control System

**Definition 2.4 (Parameterized  $\mathcal{H}_\infty$ -Control Problem)** Find a static state-feedback controller  $K \in \mathcal{K}$  if any, such that the closed-loop system has  $\mathcal{L}_2$ -gain  $\leq 1$ , i.e.,

$$\int_0^T \|z(t)\|^2 dt \leq \int_0^T \|w(t)\|^2 dt \quad (7)$$

for all  $T \in \mathbf{R}^+$  and  $\theta \in \Theta$ .

From the discussion in the last subsection, to solve the  $\mathcal{H}_\infty$ -control problem, one needs to find a parameter-dependent state-feedback controller  $u = F(x, \theta) \in \mathcal{K}$  such that the closed loop system is dissipative with respect to the supply rate  $\|w\|^2 - \|z\|^2$ ; in particular, it is sufficient that there exists a storage function  $V : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  continuously differentiable with respect to both  $\theta$  and  $x$  such that with the feedback  $F(x, \theta)$ , the following dissipation inequality holds:

$$\dot{V}(x, \theta) + \|z(t)\|^2 - \|w(t)\|^2 \leq 0 \quad (8)$$

for all  $w \in \mathcal{L}_2[0, \infty)$  and  $\theta \in \Theta$ .

Thus, the  $\mathcal{H}_\infty$ -control problem has solution if for all  $\theta \in \Theta$ , the following inequality holds.

$$\min_{u \in \mathcal{K}} \sup_{w \in \mathcal{L}_2[0, \infty)} \left\{ \frac{\partial V}{\partial x}(x, \theta) (f(x, u, \theta) + g(x, \theta)w) + \|z(t)\|^2 - \|w(t)\|^2 \right\} \leq 0 \quad (9)$$

Suppose  $R(x, \theta) := I - k^T(x, \theta)k(x, \theta) > 0$  for all  $\theta \in \Theta$  and  $x \in \mathbf{R}^n$ , then the standard manipulation in proposition 2.3 shows that the optimization problem on the left hand side of (9) is

$$\begin{aligned} & \min_{u \in \mathcal{K}} \sup_{w \in \mathcal{L}_2[0, \infty)} \left\{ \frac{\partial V}{\partial x}(x, \theta) (f(x, u, \theta) + g(x, \theta)w) + \|z(t)\|^2 - \|w(t)\|^2 \right\} \\ &= \min_{u \in \mathcal{K}} \left\{ \frac{\partial V}{\partial x}(x, \theta) (f(x, u, \theta) + g(x, \theta)R^{-1}(x, \theta)k^T(x, \theta)h(x, u, \theta)) + \right. \\ & \quad \left. \frac{1}{4} \frac{\partial V}{\partial x}(x, \theta) g(x, \theta) R^{-1}(x, \theta) g^T(x, \theta) \frac{\partial V^T}{\partial x}(x, \theta) + \right. \end{aligned}$$

$$+(h(x, u, \theta))^T (I - k(x, \theta)k^T(x, \theta))^{-1} (h(x, u, \theta)) \quad (10)$$

If the above minimization problem with respect to  $u \in \mathcal{K}$  has a solution  $u = F(x, \theta)$ , and the minimum of (10) is less than or equal to 0 for all  $x \in \mathbf{R}^n$ , then the state feedback  $\mathcal{H}_\infty$ -control problem has solution. This observation is summarized as the following proposition.

**Proposition 2.5** *The state feedback parameterized  $\mathcal{H}_\infty$ -control problem has a solution if there exist a  $\mathbf{C}^0$  function  $F : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^{p_2}$  with  $F(0, \theta) = 0$  for all  $\theta \in \Theta$  and a  $\mathbf{C}^1$  function  $V : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  to the following Hamilton-Jacobi inequality*

$$\begin{aligned} & \frac{\partial V}{\partial x}(x, \theta) (f(x, F(x, \theta), \theta) + g(x, \theta)R^{-1}(x, \theta)k^T(x, \theta)h(x, \theta)) + \\ & + \frac{1}{4} \frac{\partial V}{\partial x}(x, \theta) g(x, \theta) R^{-1}(x, \theta) g^T(x, \theta) \frac{\partial V^T}{\partial x}(x, \theta) + \\ & + (h(x, F(x, \theta), \theta))^T (I - k(x, \theta)k^T(x, \theta))^{-1} (h(x, F(x, \theta), \theta)) \leq 0. \end{aligned} \quad (11)$$

A parameterized state feedback controller is given by  $u = F(x, \theta)$ .

The parameterized  $\mathcal{H}_\infty$ -control scheme can be conveniently implemented once the control action  $u$  has access to the parameters during system operation. However, in many applications, the parameters are not available to the control action. Therefore, one needs to design a controller which combines the parameter-dependent controller and parameter adjustment mechanism. This results in the adaptive control scheme, which is the focus of the next few sections. In the adaptive  $\mathcal{H}_\infty$ -control, a very useful technique is parameter projection. As it is also of interest in its own right, the next section is entirely devoted to the treatment of projection techniques, including vector projection and direct parameter projection.

### 3 Vector and Parameter Projection Techniques

In this section, we deal with the vector and direct parameter projection techniques using the techniques from nonsmooth analysis and viability theory [2, 31]. Both projection techniques will play a very important role in the adaptive  $\mathcal{H}_\infty$ -control design. The material presented in this section is also of interest in its own right.

#### 3.1 Invariance and Contingent Cone

Consider a differential equation:

$$\dot{x} = f(x, t) \quad (12)$$

where  $f : \mathbf{R}^n \times \mathbf{R}^+ \rightarrow \mathbf{R}^n$  is continuous in  $x$  and measurable in  $t$ . Suppose for all  $x_0 \in \mathbf{R}^n$ , the differential equation has a unique solution starting from  $x(0) = x_0$  defined for  $t \in \mathbf{R}^+$ . A set  $\mathbf{X} \subset \mathbf{R}^n$  is a invariant set of (12), if for all  $x_0 \in \mathbf{X}$ , its solution will stay in the set  $\mathbf{X}$  for all  $t$ . It is known that  $\mathbf{X}$  is an invariant set, so is its closure  $\overline{\mathbf{X}}$ .

Given a compact set  $\mathbf{K} \subset \mathbf{R}^n$ , we next examine when the set is invariant for the differential equation (12). The invariance is characterized in terms of contingent cones [2]. The contingent cone to  $\mathbf{K}$  is well defined as a set-valued map  $T_{\mathbf{K}} : \mathbf{K} \rightsquigarrow X$ :

$$T_{\mathbf{K}}(x) := \{v \mid \liminf_{h \rightarrow 0^+} \frac{d_{\mathbf{K}}(x + hv)}{h} \leq 0\}. \quad (13)$$

where  $d_{\mathbf{K}}(x) := \inf_{z \in \mathbf{K}} \|x - z\|$ . For all  $x \in \mathbf{K}$ , the value  $T_{\mathbf{K}}(x)$  is a closed cone. In this following, we are concerned with only the convex sets.

Suppose  $\mathbf{K}$  is convex and  $0 \in \text{INT}(\mathbf{K})$ . The Minkowski function of  $\mathbf{K}$  is defined as (see, e.g., [21])

$$\Psi_{\mathbf{K}}(x) = \inf\{\lambda \in \mathbf{R}^+ : x \in \lambda\mathbf{K}\};$$

e.g., if a convex set is induced by a set of linear inequalities:

$$\mathbf{K} = \{x \in \mathbf{R}^n : A_i x \leq 1, i = 1, \dots, s\}, \quad (14)$$

then

$$\Psi_{\mathbf{K}}(x) := \max_{1 \leq i \leq s} \{A_i x\}.$$

Given  $r \geq 0$ , we define the set  $\mathbf{K}_r$  as

$$\mathbf{K}_r := \{x \in \mathbf{R}^n : \Psi_{\mathbf{K}}(x) \leq r\}.$$

Then  $\mathbf{K} = \mathbf{K}_1$ . The Minkowski functions of convex sets are convex, but not differentiable in general [21]. The contingent cone to a convex set can be represented in terms of the subgradient of its Minkowski function:

$$T_{\mathbf{K}}(x) = \begin{cases} \mathbf{R}^n & \text{if } \Psi_{\mathbf{K}}(x) < 1; \\ \partial\Psi_{\mathbf{K}}(x) & \text{if } \Psi_{\mathbf{K}}(x) = 1. \end{cases}$$

where  $\partial$  denotes subgradient [31]. The contingent cone to a convex set is a lower semi-continuous set-valued map on  $\mathbf{K}$  with closed convex value [2]. The following lemma provides a more explicit representation of contingent cones to convex sets [31].

**Lemma 3.1** *If  $\mathbf{K}$  is convex, then*

$$T_{\mathbf{K}}(x) = \overline{\{y : \exists t > 0 : x + ty \in \mathbf{K}\}}.$$

Another useful notion related to a convex set is its normal cone. If  $\mathbf{K} \subset \mathbf{R}^n$  is convex, then we can define its normal cone as follows:

$$N_{\mathbf{K}}(x) = \{z \in \mathbf{R}^n : y^T z \leq 0, \forall y \in T_{\mathbf{K}}(x)\}$$

Therefore,  $N_{\mathbf{K}}(x) = \{0\}$  if  $x \in \text{INT}(\mathbf{K})$ .

To conclude this subsection, we give following result about the invariance [34, 19].



**Proposition 3.2** *Given a convex compact set  $\mathbf{K} \subset \mathbf{R}^n$ , it is an invariant set of differential equation (12) if and only if for all  $x \in \mathbf{K}$ ,*

$$f(x, t) \in T_{\mathbf{K}}(x) \quad (15)$$

for all  $t \in \mathbf{R}^+$ .

Therefore, the invariance of a set can be characterized by its contingent cone. Given a convex and compact set  $\mathbf{K} \subset \mathbf{R}^n$ , the solutions of differential equation (12) are not necessarily always constrained inside the set  $\mathbf{K}$ . However, in adaptive control problems, we usually require some parameters, which are governed by differential equations, stay inside given sets during the evolution (see Sections 5 and 6), and some properties still be satisfied. In the following two subsections, we will introduce two projection methods to achieve this goal.

### 3.2 Vector Projection

Vector projection technique is introduced by Rosen [32] for constrained optimization (see also [21]). It is also widely used in the adaptive parameter estimation and adaptive control (see e.g., [10, 29, 22, 11]). In the following, it is generalized to a non-smooth case using the techniques reviewed in the last subsection.

Consider the convex and compact set  $\mathbf{K} \subset \mathbf{R}^n$  and its Minkowski function  $\Psi_{\mathbf{K}}$ . We first define the projection of a vector  $y \in \mathbf{R}^n$  at a point  $x \in \mathbf{R}^n$  on the contingent cone  $T_{\mathbf{K}}(x)$  as follows:

$$\mu_{\mathbf{K}}(x, y) = \begin{cases} y & \text{if } x \in \text{INT}(\mathbf{K}) \text{ or } y \in T_{\mathbf{K}_r}(x); \\ 0 & \text{if } y \in N_{\mathbf{K}_r}(x); \\ y^T v v & \text{otherwise, where } v = \arg \max \{y^T v_0 | v_0 \in T_{\mathbf{K}_r}(x), \|v_0\| = 1\}. \end{cases} \quad (16)$$

where  $r = \Psi_{\mathbf{K}}(x)$ . We have the following theorem.

**Theorem 3.3** *Given a convex and compact set  $\mathbf{K} \subset \mathbf{R}^n$ , the projection is defined by (16), then we have the following assertions:*

- (i)  $\mu_{\mathbf{K}}(x, y) \in T_{\mathbf{K}}(x)$  for all  $x \in \mathbf{K}$  and  $y \in \mathbf{R}^n$ .
- (ii)  $\|\mu_{\mathbf{K}}(x, y)\| \leq \|y\|$  for all  $x \in \mathbf{K}$  and  $y \in \mathbf{R}^n$ .
- (iii)  $(x - x^*)^T (\mu_{\mathbf{K}}(x, y) - y) \leq 0$  for  $x^* \in \mathbf{K}$ ,  $x \in \mathbf{K}$ , and  $y \in \mathbf{R}^n$ .
- (iv) Consider the system (12) under the above projection:

$$\dot{x} = \mu_{\mathbf{K}}(x, f(x, t)); \quad (17)$$

then the set  $\mathbf{K}$  is an invariant set for the projected system.

**Proof** Given  $x \in \mathbf{K}$  and  $y \in \mathbf{R}^n$ .

(i) If  $x \in \text{INT}(\mathbf{K})$  or  $y \in T_{\mathbf{K}}(x)$ , then  $\mu_{\mathbf{K}}(x, y) = y \in T_{\mathbf{K}}(x)$ ; if  $y \in N_{\mathbf{K}_r}(x)$ , then  $\mu_{\mathbf{K}}(x, y) = 0 \in T_{\mathbf{K}}(x)$ . We only need to show the last case. In fact, as  $v \in T_{\mathbf{K}}(x)$  and  $y^T v \geq 0$ , then  $\mu_{\mathbf{K}}(x, y) = (y^T v) v \in T_{\mathbf{K}}(x)$ .

(ii) As the inequality is satisfied trivially in the first two cases, we only need to show the case when  $\mu_{\mathbf{K}}(x, y) = y^T v v$ . Indeed,  $\|\mu_{\mathbf{K}}(x, y)\| = \|y^T v\| \leq \|y\| \|v\| = \|y\|$ .

(iii) If  $x \in \text{INT}(\mathbf{K})$  or  $y \in T_{\mathbf{K}}(x)$ , then  $\mu_{\mathbf{K}}(x, y) = y$ , so  $(x - x^*)^T(\mu_{\mathbf{K}}(x, y) - y) = 0$ . Now we consider the other cases.

Notice that, for all  $x^* \in \mathbf{K}$ , as  $x + (x^* - x) = x^* \in \mathbf{K}$ , then  $x^* - x \in T_{\mathbf{K}}(x)$  by Lemma 3.1. So if  $y \in N_{\mathbf{K}}(x)$ , then by the definition of normal cone,

$$(x - x^*)^T(\mu_{\mathbf{K}}(x, y) - y) = (x^* - x)^T y \leq 0.$$

Otherwise, let's consider the case when  $\mu_{\mathbf{K}}(x, y) = y^T v v$ . We first show that  $y - \mu_{\mathbf{K}}(x, y) \in N_{\mathbf{K}}(x)$ .

From the definition of the projection,  $v = \arg \max\{y^T v_0 | v_0 \in T_{\mathbf{K}_r}(x), \|v_0\| = 1\}$ . Therefore,

$$y^T v v = \arg \min_{z \in T_{\mathbf{K}}(x)} \|y - z\|$$

By the use of Theorem 1 in [21, p.69], we have

$$(y - y^T v v)^T(z - y^T v v) \leq 0 \quad (18)$$

for all  $z \in T_{\mathbf{K}}(x)$ . On the other hand, as  $T_{\mathbf{K}}(x)$  is a convex cone, so for all  $u \in T_{\mathbf{K}}(x)$ , then  $u + y^T v v \in T_{\mathbf{K}}(x)$ ; so (18) implies

$$(y - y^T v v)^T u = (y - y^T v v)^T(u + y^T v v - y^T v v) \leq 0.$$

Therefore,  $y - \mu_{\mathbf{K}}(x, y) \in N_{\mathbf{K}}(x)$ . Again by the definition of normal cone,

$$(x - x^*)^T(\mu_{\mathbf{K}}(x, y) - y) = (x^* - x)^T(y - \mu_{\mathbf{K}}(x, y)) \leq 0.$$

(iv) As  $\mathbf{K}$  is convex, then its Minkowski function  $\Psi_{\mathbf{K}}$  is convex, so it is absolutely continuous. Now for any absolutely continuous function  $x(t)$  that is a solution of (17) with  $x(0) \in \mathbf{K}$ , then  $\eta(t) := \Psi_{\mathbf{K}}(x(t))$  is also absolutely continuous. It is sufficient to show  $x(t) \in \mathbf{K}$ . Indeed, if it is not true, then there exists  $T > 0$ , such that  $x(T) \in \mathbf{R}^n \setminus \mathbf{K}$ ; so  $\Psi_{\mathbf{K}}(x(T)) > 0$ . Suppose  $T_0 < T$  is such that

$$T_0 = \inf\{t \geq 0 : x(t) \in \mathbf{R}^n \setminus \mathbf{K}\}$$

Therefore,  $x(T_0) \in \mathbf{K}$  as  $\mathbf{K}$  is compact, so  $\Psi_{\mathbf{K}}(x(T_0)) \leq 1$ . Let  $t \in (T_0, T)$  be a point on  $\mathbf{R}^+$  where both  $\dot{x}(t)$  and  $\frac{\partial \Psi_{\mathbf{K}}}{\partial x}(x(t))$  exist, then there exists  $\epsilon(h)$  with  $\lim_{h \rightarrow 0} \frac{\epsilon(h)}{h} = 0$  such that

$$x(t+h) = x(t) + h\dot{x}(t) + \epsilon(h).$$

Then

$$\dot{\eta}(t) = \lim_{h \rightarrow 0^+} \frac{\Psi_{\mathbf{K}}(x(t) + h\dot{x}(t) + \epsilon(h)) - \Psi_{\mathbf{K}}(x(t))}{h} = \frac{\partial \Psi_{\mathbf{K}}}{\partial x}(x(t))\dot{x}(t) = \frac{\partial \Psi_{\mathbf{K}}}{\partial x}(x(t))\mu_{\mathbf{K}}(x, f(x, t))$$

Notice that  $\frac{\partial \Psi_{\mathbf{K}}}{\partial x}(x(t)) \in N_{\mathbf{K}_r}(x(t))$  where  $r = \Psi_{\mathbf{K}}(x(t))$ ; by the same argument as (i), we can show  $\mu_{\mathbf{K}}(x, f(x, t)) \in T_{\mathbf{K}_r}(x(t))$ . Thus,

$$\dot{\eta}(t) = \frac{\partial \Psi_{\mathbf{K}}}{\partial x}(x(t))\mu_{\mathbf{K}}(x, f(x, t)) \leq 0.$$

almost everywhere on  $[T_0, T]$ . Thus,

$$0 < \Psi_{\mathbf{K}}(x(T)) - \Psi_{\mathbf{K}}(x(T_0)) = \eta(T) - \eta(T_0) = \int_{T_0}^T \dot{\eta}(t) dt \leq 0,$$

which is a contradiction. Therefore,  $\mathbf{K}$  is an invariant set.  $\square$

Notice that in the above theorem, (iv) is not the conclusion of (i), as we don't assume the solutions of (17) are unique for  $x(0) \in \mathbf{K}$  (see Proposition 3.2). In the proof of (iv), the projection property of a vector outside  $\mathbf{K}$  is used; it is remarked that the conclusion is still true if the projection (3.3) of a vector is defined onto the exterior contingent cones instead of  $T_{\mathbf{K}_r}(x)$  with  $r > 1$  (see [2, Definition 5.1.1]). (iii) is a useful property for the adaptive control design. It is also noticed that the right-hand side of (17) is not necessarily continuous, even if  $f$  is continuous. In [29], with some relaxation, the authors define a projection which is Lipschitzian and guarantees the projected system to have an invariant set larger than the parameter set.

In fact, given  $\epsilon > 0$ , define a projection  $\mu_{\mathbf{K}}^\epsilon : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  as follows:

$$\mu_{\mathbf{K}}^\epsilon(x, y) = \begin{cases} y & \text{if } x \in \text{INT}(\mathbf{K}) \text{ or } y \in T_{\mathbf{K}_r}(x); \\ 0 & \text{if } y \in N_{\mathbf{K}_r}(x); \\ y - \rho(x)v^T y v & \text{otherwise, where } v = \arg \max\{y^T v_0 | v_0 \in N_{\mathbf{K}_r}(x), \|v_0\| = 1\}. \end{cases} \quad (19)$$

where  $r = \Psi_{\mathbf{K}}(x)$  and  $\rho(x) = (\Psi_{\mathbf{K}}(x) - 1)/\epsilon$ . Using the similar arguments as in the proof of Theorem 3.3, one can show the following assertion regarding the above projection, which generalizes [29, Lemma (103)] to the nonsmooth case.

**Lemma 3.4** *Given a convex and compact set  $\mathbf{K} \subset \mathbf{R}^n$ , the projection is defined by (19), then we have the following facts:*

- (i)  $\|\mu_{\mathbf{K}}^\epsilon(x, y)\| \leq \|y\|$  for all  $x \in \mathbf{K}_{1+\epsilon}$  and  $y \in \mathbf{R}^n$ .
- (ii)  $(x - x^*)^T(\mu_{\mathbf{K}}^\epsilon(x, y) - y) \leq 0$  for  $x^* \in \mathbf{K}$ ,  $x \in \mathbf{K}_{1+\epsilon}$ , and  $y \in \mathbf{R}^n$ .
- (iii) The projected system of (12) under the projection (19):  
 $\dot{x} = \mu_{\mathbf{K}}^\epsilon(x, f(x, t))$

has an invariant set  $\mathbf{K}_{1+\epsilon} = \{x \in \mathbf{R}^n : \Psi_{\mathbf{K}}(x) \leq 1 + \epsilon\}$ .

It is remarked that the vector projection (19) is continuous with respect to  $y$  and  $x$  in the radial direction, but possibly discontinuous on  $x$ . However, if  $K$  is smooth enough, i.e., the Minkowski function is continuously differentiable, then  $\mu_{\mathbf{K}}^\epsilon$  is locally Lipschitzian [29].

### 3.3 Direct Parameter Projection

In the last subsection, we considered the vector projection to restrict the parameter in some convex set, i.e., the parameter is indirectly “projected” onto the set. However, the projection is generally not continuous. In the following, we will consider a direct parameter projection which is continuous.

Consider a convex and compact set  $\mathbf{K} \subset \mathbf{R}^n$ . The projection  $\Pi_{\mathbf{K}}(x)$  of a point  $x \in \mathbf{R}^n$  onto  $\mathbf{K}$  is defined as follows:

$$\Pi_{\mathbf{K}}(x) = \arg \min_{z \in \mathbf{K}} \|x - z\| \quad (20)$$

The above projection is well-defined, since  $\mathbf{K} \subset \mathbf{R}^n$  is convex and compact; in addition,  $\Pi_{\mathbf{K}}(x)$  is continuous. We first have the following characterization.

**Lemma 3.5** *Given a convex compact set  $\mathbf{K} \subset \mathbf{R}^n$ . Take  $x \in \mathbf{R}^n$ ; then  $\xi \in \mathbf{K}$  is such that  $\xi = \Pi_{\mathbf{K}}(x)$  if and only if*

$$x - \xi \in N_{\mathbf{K}}(\xi).$$

**Proof** From Theorem 1 in [21, p.69],  $\xi := \Pi_{\mathbf{K}}(x)$ , if and only if

$$(x - \xi)^T(x_0 - \xi) \leq 0 \quad (21)$$

for all  $x_0 \in \mathbf{K}$ . On the other hand, from Lemma 3.1 and the definition of normal cones (see also [31, Proposition 2G]),

$$N_{\mathbf{K}}(\xi) = \{v | v^T(x_0 - \xi) \leq 0, \forall x_0 \in \mathbf{K}\}. \quad (22)$$

Therefore, (21) holds if and only if  $x - \xi \in N_{\mathbf{K}}(\xi)$ .  $\square$

In the following, we have the following property of the direct projection.

**Proposition 3.6** *Take a convex compact set  $\mathbf{K} \subset \mathbf{R}^n$ . Then for any absolutely continuous function  $x : \mathbf{R} \rightarrow \mathbf{R}^n$ , its projection:*

$$\xi(t) := \Pi_{\mathbf{K}}(x(t))$$

*is also absolutely continuous.*

**Proof** It is sufficient to show that the map  $\Pi_{\mathbf{K}} : \mathbf{R}^n \rightarrow \mathbf{R}^n$  is Lipschitz. In fact, we will show that for all  $x, y \in \mathbf{R}^n$ ,

$$\|\Pi_{\mathbf{K}}(x) - \Pi_{\mathbf{K}}(y)\| \leq \|x - y\|. \quad (23)$$

Indeed,

$$\begin{aligned} \|x - y\|^2 &= \|\Pi_{\mathbf{K}}(x) - \Pi_{\mathbf{K}}(y) + (x - \Pi_{\mathbf{K}}(x)) - (y - \Pi_{\mathbf{K}}(y))\|^2 \\ &= \|\Pi_{\mathbf{K}}(x) - \Pi_{\mathbf{K}}(y)\|^2 + \|(x - \Pi_{\mathbf{K}}(x)) - (y - \Pi_{\mathbf{K}}(y))\|^2 + \\ &\quad -2(x - \Pi_{\mathbf{K}}(x))^T(\Pi_{\mathbf{K}}(y) - \Pi_{\mathbf{K}}(x)) - 2(y - \Pi_{\mathbf{K}}(y))^T(\Pi_{\mathbf{K}}(x) - \Pi_{\mathbf{K}}(y)) \\ &\geq \|\Pi_{\mathbf{K}}(x) - \Pi_{\mathbf{K}}(y)\|^2 \end{aligned}$$

where the inequality follows from the above lemma (see (21)), e.g.,

$$(x - \Pi_{\mathbf{K}}(x))^T(\Pi_{\mathbf{K}}(y) - \Pi_{\mathbf{K}}(x)) \leq 0.$$

Then the conclusion follows.  $\square$

The direct parameter projection has the following property which is useful in the adaptive  $\mathcal{H}_{\infty}$ -control problem.

**Theorem 3.7** *Given the convex and compact  $\mathbf{K}$ . Then for any absolute continuous function  $x : \mathbf{R}^+ \rightarrow \mathbf{R}^n$  with  $x(0) \in \mathbf{K}$ , and the projection:  $\xi(t) = \Pi_{\mathbf{K}}(x(t))$ , the following inequality holds:*

$$\int_0^T (\xi(t) - x^*)^T(\dot{\xi}(t) - \dot{x}(t))dt \leq 0 \quad (24)$$

for all  $x^* \in \mathbf{K}$ .

**Proof** For the given function  $x(t)$ , let

$$T_0 := \inf\{t \geq 0 : x(t) \in \mathbf{R}^n \setminus \mathbf{K}\},$$

otherwise if the above infimum does not exist, then let  $T_0 = \infty$ . Therefore, if  $t \leq T_0$ , then  $\xi(t) = x(t)$ . The relation (24) holds trivially:

$$\int_0^T (\xi(t) - x^*)^T (\dot{\xi}(t) - \dot{x}(t)) dt = 0$$

for all  $T \leq T_0$ .

Now it is sufficient to show when  $T_0 < \infty$ , and  $T > T_0$ , the inequality (24) is satisfied. Therefore,

$$\begin{aligned} \int_0^T (\xi(t) - x^*)^T (\dot{\xi}(t) - \dot{x}(t)) dt &= \int_{T_0}^T (\xi(t) - x^*)^T d(\xi(t) - x(t)) \\ &= (\xi(t) - x(t))^T (\xi(t) - x^*) \Big|_{T_0}^T - \int_{T_0}^T (\xi(t) - x(t))^T d\xi(t) \\ &= (\xi(T) - x(T))^T (\xi(T) - x^*) - \int_{T_0}^T (\xi(t) - x(t))^T \dot{\xi}(t) dt \end{aligned}$$

where the last equality holds because  $\xi(T_0) = x(T_0)$ .

Notice that from Lemma 3.5,

$$x(t) - \xi(t) \in N_{\mathbf{K}}(\xi(t)) \tag{25}$$

for all  $t \in [T_0, T]$ .

Now  $\xi(T) + (x^* - \xi(T)) = x^* \in \mathbf{K}$ , then by Lemma 3.1,

$$x^* - \xi(T) \in T_{\mathbf{K}}(\xi(T)).$$

Thus, (25) implies

$$(\xi(T) - x(T))^T (\xi(T) - x^*) = (x(T) - \xi(T))^T (x^* - \xi(T)) \leq 0.$$

On the other hand, if  $\xi(t) \in \mathbf{K}$  is differentiable at  $t \in (T_0, T)$ , there exists a positive sequence  $\{h_n\}$  with  $h_n \rightarrow 0$  as  $n \rightarrow \infty$  such that  $\xi(t + h_n) \in \mathbf{K}$ ; denote

$$d_n(t) := \frac{\xi(t + h_n) - \xi(t)}{h_n},$$

then  $d_n(t) \rightarrow \dot{\xi}(t)$  as  $n \rightarrow \infty$ . Since  $\xi(t) + h_n d_n(t) = \xi(t + h_n) \in \mathbf{K}$ ,  $d_n(t) \in T_{\mathbf{K}}(\xi(t))$  by Lemma 3.1. Therefore,  $\dot{\xi}(t) = \lim_{n \rightarrow \infty} d_n(t) \in T_{\mathbf{K}}(\xi(t))$  as  $T_{\mathbf{K}}(\xi(t))$  is a closed cone. Thus, (25) implies

$$(\xi(t) - x(t))^T \dot{\xi}(t) = -(x(t) - \xi(t))^T \dot{\xi}(t) \geq 0.$$

for all  $t \in [T_0, T]$ . Therefore,

$$\int_0^T (\xi(t) - x^*)^T (\dot{\xi}(t) - \dot{x}(t)) dt = (\xi(T) - x(T))^T (\xi(T) - x^*) - \int_{T_0}^T (\xi(t) - x(t))^T \dot{\xi}(t) dt \leq 0$$

□

In the above proof, we have also proved the following useful result.

**Corollary 3.8** *Let  $\xi : \mathbf{R}^+ \rightarrow \mathbf{R}^n$  be an absolutely continuous function. Give a compact and convex set  $\mathbf{K} \in \mathbf{R}^n$ ; if  $\xi(t) \in \mathbf{K}$  for all  $t \in \mathbf{R}^n$ , then*

$$\dot{\xi}(t) \in T_{\mathbf{K}}(\xi(t))$$

*for almost all  $t \in \mathbf{R}^n$ .*

## 4 Adaptive $\mathcal{H}_\infty$ -Control Problem

We will consider the adaptive  $\mathcal{H}_\infty$ -control problem in the next few sections. The uncertain nonlinear system to be considered is governed by the following parameterized dynamical equation:

$$G(\theta) : \begin{cases} \dot{x} = f(x, \theta) + g_1(x, \theta)w + g_2(x, \theta)u \\ z = h(x) + k_1(x)w + k_2(x)u \\ y = h_2(x) + k_{21}(x)w + k_{22}(x)u \end{cases} \quad (26)$$

where  $\theta$  is a  $r$ -dimensional vector of unknown constant parameters with  $\theta \in \Theta \in \mathbf{R}^r$ . In the adaptive control problem, we will consider the parameter-dependence in the following fashion:

$$f(x, \theta) = f_0(x) + \sum_{i=1}^r \theta_i f_i(x)$$

$$g_j(x, \theta) = g_{j0}(x) + \sum_{i=1}^r \theta_i g_{ji}(x), \quad j = 1, 2.$$

It is assumed that  $f_i, g_{ji}, h, k_j \in \mathbf{C}^0$ ,  $f_i(0) = 0, h(0) = 0$ , and  $R(x) := I - k_1^T(x)k_1(x) > 0$  for all  $x \in \mathbf{R}^n$ ;  $x, w, u, z$ , and  $y$  are state, exogenous disturbance, control input, regulated output, and measured output with dimensions  $n, p_1, p_2, q$ , and  $n + p_1$ , respectively.

The objective of the adaptive  $\mathcal{H}_\infty$ -controller design is to attenuate the impact of the exogenous disturbance  $w$  and the error induced by the initial guess of the parameter. The impact is measured by the output  $z$ , and both signals  $w$  and  $z$  are measured by their  $\mathcal{L}_2$ -norms. The adaptive controllers to be sought have the following form.

$$K : \begin{cases} \dot{p} = \phi(p, y, u) \\ u = \kappa(p)y \end{cases} \quad (27)$$

where  $p \in \mathbf{R}^r$  is the estimation of the real parameter  $\theta$ ,  $\phi \in \mathbf{C}^0$ , and  $\dot{p} = \phi(p, y, u)$  is the parameter update law. For fixed  $p$ ,  $u = \kappa(p)y$  is a I/O map from  $y$  to  $u$ ; it is taken as a (possibly modified) gain-scheduled controller in the sequel. An adaptive  $\mathcal{H}_\infty$  control system is illustrated in Figure 2. The precise statement of the adaptive control problem is given next.

**Definition 4.1 (Adaptive  $\mathcal{H}_\infty$ -Control Problem)** Suppose  $\epsilon > 0$  is given. The adaptive  $\mathcal{H}_\infty$ -control design is to seek a controller (27) such that the resulting closed loop system with  $x(0) = 0$  satisfies

$$\int_0^T \|z(t)\|^2 dt \leq \int_0^T \|w(t)\|^2 dt + \epsilon \quad (28)$$

for all  $T \in \mathbf{R}^+$ ,  $w \in \mathcal{L}_2[0, \infty)$ ,  $p(0) \in \Theta$ , and  $\theta \in \Theta$ .

**Remark 4.2** The above performance (28) is weaker than the original  $\mathcal{H}_\infty$ -performance:

$$\int_0^T \|z(t)\|^2 dt \leq \int_0^T \|w(t)\|^2 dt.$$

It will be seen that the extra  $\epsilon$ -term in (28) captures the transient effect due to the mismatching between the initial guess ( $p(0)$ ) and the true value of the parameter  $\theta$ . However, this effect could be made arbitrarily small by suitable design of the controller.

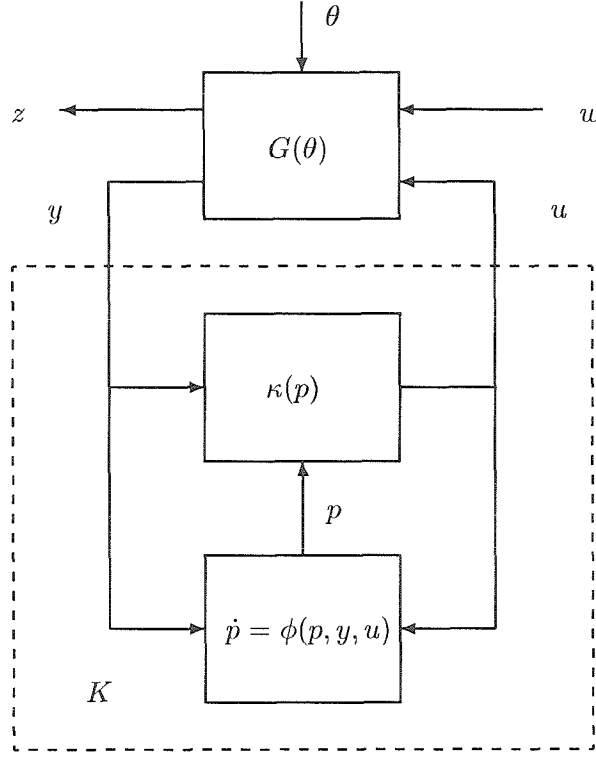


Figure 2: Adaptive  $\mathcal{H}_\infty$ -Control System

### Adaptive $\mathcal{H}_\infty$ -Control with Full Information Feedback

Consider the system (26). In this paper, we consider full-information feedback, in which case the measured output is  $y = \begin{bmatrix} x \\ w \end{bmatrix}$ , i.e., both  $x$  and  $w$  are available to the control input  $u$ .

It is known that if the parameter  $\theta$  is known, then from Proposition 2.5, the  $\mathcal{H}_\infty$ -control problem has a state-feedback solution if there exist a continuous function  $F : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^{p_2}$  and a non-negative function  $V : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  which is positive definite with respect to  $x$  such that the following parameter dependent Hamilton-Jacobi inequality is satisfied for all  $\theta \in \Theta$ :

$$\begin{aligned}
& \frac{\partial V}{\partial x}(x, \theta) (f(x, \theta) + g_1(x, \theta) R^{-1}(x) k_1^T(x) h(x) + g_2(x) F(x, \theta)) + \\
& + \frac{1}{4} \frac{\partial V}{\partial x}(x, \theta) (g_1(x, \theta) R^{-1}(x) g_1^T(x, \theta) + \\
& + (h^T(x) + k_2(x) F(x, \theta)) (I - k_1(x) k_1^T(x))^{-1} (h(x) + k_2(x) F(x, \theta))) \leq 0;
\end{aligned} \tag{29}$$

the parameter-dependent static state-feedback,

$$u = F(x, \theta), \tag{30}$$

is an  $\mathcal{H}_\infty$ -controller for the parameter-dependent nonlinear system; and the closed-loop system satisfies (7). However, if we don't know the parameter before the system is in operation, we need to design an adaptive mechanism to estimate the parameter on-line and use the estimated parameter to adjust the necessary control action; in which case, the controller

$$u = \psi(x, w, p)$$

is used instead, where  $\psi$  is some modification of  $F$  in (30),  $p$  is an estimation of  $\theta$ , and its update law has the following general form:

$$\dot{p} = \phi(p, x, w, u)$$

The adaptive control problem is stated precisely as follows:

**Definition 4.3 (Full Information Feedback Solution)** *Given  $\epsilon > 0$ ; find an adaptive controller*

$$K_{FI} : \begin{cases} \dot{p} = \phi(p, x, w, u) \\ u = \psi(x, w, p) \end{cases} \quad (31)$$

where  $\phi, \psi$  are suitable functions, such that the closed loop system with  $x(0) = 0$  satisfies (28) for all  $T \in \mathbf{R}^+$ ,  $w \in \mathcal{L}_2[0, \infty)$ ,  $p(0) \in \Theta$ , and  $\theta \in \Theta$ .

In the next section, we will give solutions to the above FI adaptive  $\mathcal{H}_\infty$ -control problem with full information feedback.

## 5 Solutions to Adaptive $\mathcal{H}_\infty$ -Control Problem

In this section, we will examine the solution to the adaptive  $\mathcal{H}_\infty$ -control problem with full information (FI) feedback. Consider system (26), we make the following assumption to simplify the development.

**Assumption 5.1** *Consider the system (26).*

[A1] *The parameter set  $\Theta$  is convex and compact, and  $0 \in \text{INT}(\Theta)$ .*

$$[A2] \quad y = \begin{bmatrix} x \\ w \end{bmatrix}.$$

[A3]  *$k_1(x) = 0$  and  $k_2(x) = 0$  for all  $x \in \mathbf{R}^n$ .*

Assumption [A2] just restates the full information problem. [A3] means that the regulated output is independent of direct disturbance and control inputs. This assumption is just for the sake of simplicity; it can be replaced by the standard assumption in the  $\mathcal{H}_\infty$ -control problem [8, 18] in most of the derivations in the following, i.e.,

$$[A3'] \quad k_1(x) = 0 \text{ and } k_2^T(x) \begin{bmatrix} h(x) & k_2(x) \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \text{ for all } x \in \mathbf{R}^n.$$

**Remark 5.2** *It will be seen that for the parameter-dependent system (26) with above assumptions, if the multiplier  $g_1(x, \theta)$  of the disturbance  $w$  is independent of  $\theta$ , then only the state information, in stead of full information, is needed to construct the adaptive  $\mathcal{H}_\infty$ -control law.*



## 5.1 Adaptive $\mathcal{H}_\infty$ -Control, Dissipativity, and Minimax Optimization

Our starting point of adaptive  $\mathcal{H}_\infty$ -control design is the condition for the (parameterized)  $\mathcal{H}_\infty$ -control problem to have solutions when the parameter  $\theta$  is known. From Section 2.2, the parameterized  $\mathcal{H}_\infty$ -control problem has solutions if there exists a nonnegative function  $V : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  which is positive definite with respect to  $x$  such that the HJI (29) is satisfied. It can be verified that (29) holds if the following parameter dependent Hamilton-Jacobi inequality is satisfied:

$$\frac{\partial V}{\partial x}(x, \theta) f(x, \theta) + \frac{1}{4} \frac{\partial V}{\partial x}(x, \theta) (g_1(x, \theta) g_1^T(x, \theta) - g_2(x, \theta) g_2^T(x, \theta)) \frac{\partial V}{\partial x}(x, \theta) + h^T(x) h(x) \leq 0 \quad (32)$$

and the parameter-dependent  $\mathcal{H}_\infty$ -controller<sup>1</sup> is

$$u = -\frac{1}{4} g_2(x, \theta) \frac{\partial V}{\partial x}(x, \theta). \quad (33)$$

Moreover, the closed loop system with the above controller is dissipative with respect to the supply rate  $\|w\|^2 - \|z\|^2$ , and the function  $V$  is a storage function for the closed-loop system satisfying (8), i.e.,

$$\dot{V}(x, \theta) \leq \|w\|^2 - \|z\|^2.$$

However, if we don't exactly know the parameter before the system is in operation, we need to design an adaptive controller (31) to accomplish the above duty. Now the adaptive controller (31) is applied, then the resulting closed loop system (26)-(31) has state  $(x, p)$ :

$$\begin{cases} \dot{x} = f(x, \theta) + g_1(x, \theta)w + g_2(x, \theta)u \\ \dot{p} = \phi(p, x, w, u) \\ u = \psi(x, w, p) \\ z = h(x) + k_1(x)w + k_2(x)u \end{cases} \quad (34)$$

To guarantee the  $\mathcal{H}_\infty$ -performance (28) for the closed system, we need to show that the adaptive system (34) is dissipative with respect to the supply rate  $\|w\|^2 - \|z\|^2$ ; it is enough to find a storage function  $W_\theta : \mathbf{R}^n \times \mathbf{R}^r \rightarrow \mathbf{R}^+$  for each  $\theta \in \Theta$ , such that the following dissipation inequality is satisfied:

$$W_\theta(x(T), p(T)) - W_\theta(x(0), p(0)) \leq \int_0^T (\|w(t)\|^2 - \|z(t)\|^2) dt.$$

or its differential version is satisfied if  $W$  is differentiable:

$$\begin{aligned} \dot{W}_\theta(x, p) &= \frac{\partial W_\theta}{\partial x}(x, p) (f(x, \theta) + g_1(x, \theta)w + g_2(x, \theta)\psi(x, w, p)) + \frac{\partial W_\theta}{\partial p}(x, p) \phi(p, x, w, \psi(x, w, p)) \\ &\leq \|w\|^2 - \|z\|^2. \end{aligned} \quad (35)$$

for each  $\theta \in \Theta$  and  $(x, p) \in \mathbf{R}^n \times \mathbf{R}^r$ .

Another interpretation to the adaptive  $\mathcal{H}_\infty$ -control is to use Willems' Lemma (Lemma 2.2). In fact, motivated by (3), we can consider the following minimax problem

$$W_\theta^a(x, p) := \min_{u(x, w, u)} \sup_{w \in \mathcal{L}_2[0, \tau], \tau \geq 0, x(0) = x} - \int_0^\tau (\|w(t)\|^2 - \|z(t)\|^2) dt. \quad (36)$$

---

<sup>1</sup>If [A3] is replaced by [A3'], then  $u = -\frac{1}{2} g_2(x, \theta) \frac{\partial V}{\partial x}(x, \theta)$ .

If the above minimax problem has an solution for each  $\theta \in \Theta$ , i.e.,  $W_\theta^a(x, p) < \infty$  for all possible  $(x, p)$  and  $\theta \in \Theta$ , then the adaptive system has  $\mathcal{L}_2$ -gain  $\leq 1$ . The above minimax adaptive control problem is discussed in [7] with the assumption  $g_1(x, \theta) = I$  by the use of differential game theoretical methods..

However, for the adaptive  $\mathcal{H}_\infty$ -control problem, we don't need to consider the optimal solution of the above minimax problem (36), which though implies the adaptive  $\mathcal{H}_\infty$ -control solution<sup>2</sup>. Instead, we just need to find an adaptive controller such that the resulting system is dissipative, i.e., it is sufficient to find the storage function  $W_\theta : \mathbf{R}^n \times \mathbf{R}^r \rightarrow \mathbf{R}^+$  such that

$$\begin{aligned} \min_{\phi, \psi} \sup_{w \in \mathcal{L}_2[0, \infty)} \{ & \frac{\partial W_\theta}{\partial x}(x, p)(f(x, \theta) + g_1(x, \theta)w + g_2(x, \theta)\psi(x, w, p)) + \\ & + \frac{\partial W_\theta}{\partial p}(x, p)\phi(p, x, w, \psi(x, w, p)) - \|w\|^2 + \|z\|^2 \} \leq 0 \end{aligned} \quad (37)$$

for  $\theta \in \Theta$ , which is a sufficient condition for both the adaptive  $\mathcal{H}_\infty$ -control problem and the adaptive minimax problem (36) to have solutions. The above discussion is summarized in the following proposition.

**Proposition 5.3** *Given  $\epsilon > 0$ . If there exist a  $\mathbf{C}^1$  nonnegative function  $W_\theta : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  with  $W_\theta(0, p(0)) \leq \epsilon$  for each  $\theta$  and suitable functions  $\psi, \phi$  such that (35) is satisfied, then the following statements are true:*

- (i) *The adaptive  $\mathcal{H}_\infty$ -control problem has solutions, and such an adaptive  $\mathcal{H}_\infty$ -controller is given by (31);*
- (ii) *If the system is reachable from the equilibrium through input  $w$ , then the minimax adaptive control problem (36) has a solution; moreover, the optimal value function satisfies  $W_\theta^a(x, p) \leq W_\theta(x, p)$  for all possible  $(x, p)$  and  $\theta \in \Theta$ .*

In the following, we will explicitly construct storage functions such that (37) is satisfied. From the discussion in Section 2.2, suppose the Hamilton-Jacobi inequality (32) has a solution  $V(x, \theta)$ , a possible choice for the storage function is  $V(x, p)$  where the unknown parameter  $\theta$  is replaced by its estimation  $p$ . However, it does not reflect the parameter estimation nature for the update law. On the other hand, the parameter enters the system in an affine fashion. Therefore, a meaningful choice of the storage function of the adaptive control system is the one with an additional quadratic  $p$ -term:

$$W_\theta(x, p) = V(x, p) + (p - \theta)^T Q (p - \theta)$$

It is remarked that this idea was first introduced to construct Lyapunov functions for stable adaptive systems [26], and used in many adaptive control problems [15, 29, 38, 17]. For the sake of simplicity, we will assume that the function  $V : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  satisfying the the above Hamilton-Jacobi inequality be continuously differentiable with respect to both arguments.

In the next few subsections, we will give the detailed solutions to the adaptive  $\mathcal{H}_\infty$ -control problem with full information feedback in different cases when  $V$  is dependent on or independent of the parameter.

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<sup>2</sup>From Lemma 2.2, it is known that the solvability of adaptive  $\mathcal{H}_\infty$ -control problem and the solvability of minimax adaptive control problem are the same if the resulting adaptive system is reachable from  $(x, p) = (0, \theta)$ .

## 5.2 Parameter-Independent Storage Function

In this section, we mainly consider the case where there exists a positive definite function  $V : \mathbf{R}^n \rightarrow \mathbf{R}^+$  which is independent of  $\theta$  such that it satisfies the Hamilton-Jacobi inequality (32), i.e.,

$$\frac{\partial V}{\partial x}(x)f(x, \theta) + \frac{1}{4} \frac{\partial V}{\partial x}(x)(g_1(x, \theta)g_1^T(x, \theta) - g_2(x, \theta)g_2^T(x, \theta)) \frac{\partial V^T}{\partial x}(x) + h^T(x)h(x) \leq 0. \quad (38)$$

for all  $\theta \in \Theta$ .

Let  $W : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  be a positive definite function defined as

$$W(x, p) = V(x) + \alpha(p - \theta)^T(p - \theta). \quad (39)$$

where  $\alpha > 0$  can be chosen such that

$$\max_{(p, \theta) \in \Theta \times \Theta} \alpha(p - \theta)^T(p - \theta) \leq \epsilon,$$

because  $\Theta \times \Theta$  is compact. Take  $W$  as a storage function candidate of the adaptive  $\mathcal{H}_\infty$ -control system. Then

$$\begin{aligned} \dot{W}(x, p) &= \dot{V}(x) + 2\alpha(p - \theta)^T \dot{p} \\ &= \frac{\partial V}{\partial x}(x)(f(x, \theta) + g_1(x, \theta)w + g_2(x, \theta)u) + 2\alpha(p - \theta)^T \dot{p} \\ &= \frac{\partial V}{\partial x}(x)(f(x, p) + g_1(x, p)w + g_2(x, p)u) + \\ &\quad + \sum_{i=1}^r \left\{ \frac{\partial V}{\partial x}(x)(\theta_i - p_i)(f_i(x) + g_{1i}(x)w + g_{2i}(x)u) \right\} + 2\alpha(p - \theta)^T \dot{p} \end{aligned}$$

Notice that if  $p \in \Theta$ , then from the assumption (38), then

$$\frac{\partial V}{\partial x}(x)f(x, p) \leq -\left(\frac{1}{4} \frac{\partial V}{\partial x}(x)(g_1(x, p)g_1^T(x, p) - g_2(x, p)g_2^T(x, p)) \frac{\partial V^T}{\partial x}(x) + h^T(x)h(x)\right)$$

Replace the above inequality and use the completion of square, one has

$$\begin{aligned} \dot{W}(x, p) &\leq \|w(t)\|^2 - \|z(t)\|^2 + \frac{\partial V}{\partial x}(x)g_2(x, p)(u(t) + \frac{1}{4}g_2^T(x, p) \frac{\partial V^T}{\partial x}(x)) + \\ &\quad - \left\| w(t) - \frac{1}{2}g_1^T(x, p) \frac{\partial V^T}{\partial x}(x) \right\|^2 + 2\alpha(p - \theta)^T(\dot{p} - \Phi(x, w, u)), \end{aligned} \quad (40)$$

where  $\Phi : \mathbf{R}^n \times \mathbf{R}^{p_1} \times \mathbf{R}^{p_2} \rightarrow \mathbf{R}^r$  is defined as

$$\Phi(x, w, u) = \frac{1}{2\alpha} \begin{bmatrix} \frac{\partial V}{\partial x}(x)(f_1(x) + g_{11}(x)w + g_{21}(x)u) \\ \frac{\partial V}{\partial x}(x)(f_2(x) + g_{12}(x)w + g_{22}(x)u) \\ \vdots \\ \frac{\partial V}{\partial x}(x)(f_r(x) + g_{1r}(x)w + g_{2r}(x)u) \end{bmatrix}. \quad (41)$$

From (40), one has that if  $p \in \Theta$  and  $u = -\frac{1}{4}g_2^T(x, p)\frac{\partial V^T}{\partial x}(x)$ , then

$$\dot{W}(x, p) \leq \|w(t)\|^2 - \|z(t)\|^2 + 2\alpha(p - \theta)^T(\dot{p} - \Phi(x, w, u)), \quad (42)$$

Now integrate both sides of (42) from 0 to  $T$  and notice  $W(x(T), p(T)) \geq 0$  and  $x(0) = 0$ , we have

$$\begin{aligned} \int_0^T \|z(t)\|^2 dt &\leq \int_0^T \|w(t)\|^2 dt + \alpha(p(0) - \theta)^T(p(0) - \theta) + \\ &+ 2\alpha \int_0^T (p(t) - \theta)^T(\dot{p}(t) - \Phi(x(t), w(t), u(t)))dt \\ &\leq \int_0^T \|w(t)\|^2 dt + 2\alpha \int_0^T (p(t) - \theta)^T(\dot{p}(t) - \Phi(x(t), w(t), u(t)))dt + \epsilon. \end{aligned} \quad (43)$$

Therefore, if we can find a parameter update law for  $p$  such that

$$\int_0^T (p(t) - \theta)^T(\dot{p}(t) - \Phi(x(t), w(t), u(t)))dt \leq 0$$

and

$$p(t) \in \Theta, \quad \forall t \in \mathbf{R}^+,$$

then the adaptive  $\mathcal{H}_\infty$ -control problem is solved. Fortunately, we can use the projection techniques developed in Section 3 to achieve the above requirements.

**Theorem 5.4 (Adaptive  $\mathcal{H}_\infty$ -Control with Vector Projection)** *Consider the parameter-dependent system (26). Suppose there exists a non-negative function  $V : \mathbf{R}^n \rightarrow \mathbf{R}^+$  such that for each  $\theta \in \Theta$ , (38) is satisfied. Then the adaptive  $\mathcal{H}_\infty$ -control problem has a solution for any  $\epsilon > 0$ . And an adaptive control law is given by*

$$\begin{cases} \dot{p} = \mu(p, \Phi(x, w, -\frac{1}{4}g_2^T(x, p)\frac{\partial V^T}{\partial x}(x))) \\ u = -\frac{1}{4}g_2^T(x, p)\frac{\partial V^T}{\partial x}(x) \end{cases} \quad (44)$$

where  $\Phi$  is defined by (41) and  $\mu$  is the vector projection with respect to the set  $\Theta$ .

**Proof** Consider the adaptive control law (44). From Theorem 3.3, one has that the given parameter update law:

$$\dot{p} = \mu(p, \Phi(x, w, u)),$$

insures  $p(t) \in \Theta$  and

$$(p - \theta)^T(\dot{p} - \Phi(x, w, u)) = (p - \theta)^T(\mu(p, \Phi(x, w, u)) - \Phi(x, w, u)) \leq 0,$$

which implies

$$\int_0^T (p(t) - \theta)^T(\dot{p}(t) - \Phi(x(t), w(t), u(t)))dt \leq 0$$

for all  $T \in \mathbf{R}^+$ . Now apply the adaptive control law (44), we have the relation (43), which implies

$$\int_0^T \|z(t)\|^2 dt \leq \int_0^T \|w(t)\|^2 dt + \epsilon,$$

for all  $T \in \mathbf{R}^+$ . □

The direct parameter projection method can be also applied.

**Theorem 5.5 (Adaptive  $\mathcal{H}_\infty$ -Control with Direct Parameter Projection)** *Consider the parameter-dependent system (26). Suppose there exists a non-negative function  $V : \mathbf{R}^n \rightarrow \mathbf{R}^+$  such that for each  $\theta \in \Theta$ , (38) is satisfied. Then the adaptive  $\mathcal{H}_\infty$ -control problem has a solution for any  $\epsilon > 0$ . And an adaptive control law is given by*

$$\begin{cases} \dot{\pi} = \Phi(x, w, -\frac{1}{4}g_2^T(x, p)\frac{\partial V^T}{\partial x}(x)) \\ p = \Pi(\pi) \\ u = -\frac{1}{4}g_2^T(x, p)\frac{\partial V^T}{\partial x}(x) \end{cases} \quad (45)$$

where  $\Phi$  is defined by (41) and  $\Pi$  is the direct parameter projection with respect to the set  $\Theta$ .

**Proof** Consider the adaptive control law (45). Suppose  $p(t)$  for  $t \in [0, \infty)$  is generated by the resulting update law:

$$\dot{\pi} = \Phi(x, w, u),$$

and

$$p(t) = \Pi(\pi(t));$$

then from Theorem 3.7, one has  $p(t) \in \Theta$  and for all  $T \in \mathbf{R}^+$ ,

$$\int_0^T (p(t) - \theta)^T (\dot{p}(t) - \Phi(x(t), w(t), u(t))) dt = \int_0^T (p(t) - \theta)^T (\dot{p}(t) - \dot{\pi}(t)) dt \leq 0.$$

Now apply the adaptive control law (45), we have the relation (43), which implies

$$\int_0^T \|z(t)\|^2 dt \leq \int_0^T \|w(t)\|^2 dt + \epsilon,$$

for all  $T \in \mathbf{R}^+$ . □

**Remark 5.6** *It is interesting to compare Theorem 5.5 with the sufficient condition result for the minimax adaptive problem in [7]. As in the above theorem 5.5, the sufficient condition in [7] for the minimax adaptive control problem to have solution is that there exists a non-negative function  $V : \mathbf{R}^n \rightarrow \mathbf{R}^+$  independent of the parameter such that the Hamilton-Jacobi inequality (38) is satisfied; and the control action  $u$  is also obtained by the parameter projection. However, the implications of the function  $V(x)$  in the two papers are different. In this paper,  $V(x)$  is just the storage function of the parameterized  $\mathcal{H}_\infty$ -control system, but not the resulting adaptive  $\mathcal{H}_\infty$ -control system. In [7],  $V(x)$  is the storage function of the resulting minimax adaptive control system, i.e., an upper bound of the value function. Due to the nice structure of the adaptive storage function constructed in (39), the adaptive controller in this paper is much simpler than that given in [7]. In addition, the use and implications of parameter projection in [7] and this paper are different; consequently, the way of parameter projection in [7] is dependent on state, while the projection in this paper is constant in this sense.*

It is noted that the direct parameter projection guarantees that the adaptive control law (45) is continuous, while the adaptive control law (44) using vector projection is not. The latter control law can be made continuous using the vector projection defined by (16) under some smoothness assumption.

### 5.3 Parameter-Dependent Storage Function

In this section, we consider the case where there exists a positive definite function  $V : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  which satisfies the Hamilton-Jacobi inequality (32) for all  $\theta \in \Theta$ . Under this assumption, we will see how to design the adaptive  $\mathcal{H}_\infty$  controller.

When  $V$  depends on the parameter, with the storage function  $W(x, p) = V(x, p) + \alpha(p - \theta)^T(p - \theta)$ , the adaptive controller obtained in the last section is not sufficient to guarantee the parameter-dependent system satisfies the dissipation inequality (42):

$$\dot{W}(x, p) \leq \|w(t)\|^2 - \|z(t)\|^2.$$

Because the additional term  $\frac{\partial V}{\partial p}(x, p)\dot{p}$  will appear on the right hand side of (42). However, if during the operation, if

$$\frac{\partial V}{\partial p}(x, p)\dot{p} \leq 0, \quad (46)$$

then this problem can still be solved. Notice that under either vector or direct parameter projection,  $\dot{p} \in T_\Theta(p)$  (the former case is guaranteed by the definition; for the latter case see Corollary 3.8). Therefore, if  $\frac{\partial V}{\partial p}(x, p) \in N_\Theta(p)$ , then (46) is satisfied. Now define

$$\Phi(x, w, u, p) = \frac{1}{2\alpha} \begin{bmatrix} \frac{\partial V}{\partial x}(x, p)(f_1(x) + g_{11}(x)w + g_{21}(x)u) \\ \frac{\partial V}{\partial x}(x, p)(f_2(x) + g_{12}(x)w + g_{22}(x)u) \\ \vdots \\ \frac{\partial V}{\partial x}(x, p)(f_r(x) + g_{1r}(x)w + g_{2r}(x)u) \end{bmatrix}; \quad (47)$$

we have the following assertion.

**Proposition 5.7** *Consider the parameter-dependent system (26). Suppose there exists a non-negative function  $V : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  such that for each  $\theta \in \Theta$ , (38) is satisfied, in addition, for all  $x \in \mathbf{R}^n$ ,*

$$\frac{\partial V}{\partial p}(x, p) \in N_\Theta(p) \quad (48)$$

*for  $p \in \Theta$ . Then the adaptive  $\mathcal{H}_\infty$ -control problem has solutions for any  $\epsilon > 0$ . And two such adaptive control laws are given by*

$$\begin{cases} \dot{p} = \mu(p, \Phi(x, w, -\frac{1}{4}g_2^T(x, p)\frac{\partial V^T}{\partial x}(x, p), p)) \\ u = -\frac{1}{4}g_2^T(x, p)\frac{\partial V^T}{\partial x}(x, p) \end{cases} \quad (49)$$

and

$$\begin{cases} \dot{\pi} = \Phi(x, w, -\frac{1}{4}g_2^T(x, p)\frac{\partial V^T}{\partial x}(x, p), p) \\ p = \Pi(\pi) \\ u = -\frac{1}{4}g_2^T(x, p)\frac{\partial V^T}{\partial x}(x, p) \end{cases} \quad (50)$$

where  $\Phi$  is defined in (47)  $\mu$  is the vector projection with respect to the set  $\Theta$ , and  $\Pi$  is the direct parameter projection with respect to the set  $\Theta$ .

If the storage function  $V$  is independent of the parameter on  $\Theta$ , then (48) is automatically satisfied, because  $\frac{\partial V}{\partial p}(x, p) = 0 \in N_{\Theta}(p)$ . It is remarked that the condition (48) is still very restrictive. It requires that  $V$  is independent of  $p$  on  $\text{INT}(\Theta)$ , as the normal cone  $N_{\Theta}(p) = \{0\}$  in this case. Therefore, we need to consider some other alternatives. Motivated by the treatment in [15, 17], one can use a control with the following form:

$$u = u^*(x, p) + v \quad (51)$$

where  $u^*(x, p) = -\frac{1}{4}g_2^T(x)\frac{\partial V^T}{\partial x}(x, p)$  and  $v$  is to be decided.

Now, replace  $u = u^*(x, p) + v$  in the original system (26), one has the following resulting representation of the parameterized plant:

$$G(\theta) : \begin{cases} \dot{x} &= f(x, \theta) + g_1(x, \theta)w + g_2(x, \theta)u^*(x, p) + g_2(x, \theta)v \\ z &= h(x) + k_1(x)w + k_2(x)(u^*(x, p) + v) \end{cases} \quad (52)$$

where  $\theta \in \Theta$ ,  $p \in \Theta$  is the estimated parameter of  $\theta$ , and its updated law is to be decided.

Now define a positive definite function  $W : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  as:

$$W(x, p) = V(x, p) + \alpha(p - \theta)^T(p - \theta)$$

If  $p \in \Theta$ , then

$$\begin{aligned} \dot{W}(x, p) &\leq \|w\|^2 - \|z\|^2 + 2\alpha(p - \theta)^T(\dot{p} - \Phi(x, w, u^*(x, p) + v, p)) + \\ &+ \frac{\partial V}{\partial x}(x, p)g_2(x, p)v + \frac{\partial V}{\partial p}(x, p)\dot{p} \end{aligned}$$

where  $\Phi : \mathbf{R}^n \times \mathbf{R}^{p_1} \times \mathbf{R}^{p_2} \times \Theta \rightarrow \mathbf{R}^r$  is defined as in (47). Therefore,

$$\dot{W}(x, p) \leq \|w\|^2 - \|z\|^2 \quad (53)$$

if the following conditions are satisfied:

$$\frac{\partial V}{\partial x}(x, p)g_2(x, p)v + \frac{\partial V}{\partial p}(x, p)\dot{p} \leq 0; \quad (54)$$

$$p \in \Theta; \quad (55)$$

$$(p - \theta)^T(\dot{p} - \Phi(x, w, u^*(x, p) + v, p)) \leq 0; \quad (56)$$

The condition (54) is called the matching condition that will be assumed to hold for some function  $v$ . The other conditions can be guaranteed by the use of projection to choose the update law and the control function. As discussed in the previous section, from Theorem 3.3 the conditions (55) and (56) are satisfied by choosing the parameter update law as

$$\dot{p} = \mu(p, \Phi(x, w, u^*(x, p) + v, p))$$

where  $\mu$  is the vector projection with respect to the set  $\Theta$ . Therefore, if there exists a function  $v(x, p, \delta)$  to satisfying the following inequality:

$$\frac{\partial V}{\partial x}(x, p)g_2(x, p)v + \frac{\partial V}{\partial p}(x, p)\delta \leq 0 \quad (57)$$

for  $\delta \in T_\Theta(p)$ , then (54) is satisfied with  $v(x, p, \dot{p})$  as  $\dot{p} \in T_\Theta(p)$  (see Corollary 3.8), (there is more discussion about the matching condition (57) at the end of this subsection). Thus, (53) is satisfied by the following adaptive law:

$$\begin{cases} \dot{p} = \mu(p, \Phi(x, w, u, p)) \\ u = -\frac{1}{4}g_2^T(x, p)\frac{\partial V^T}{\partial x}(x) + v(x, p, \dot{p}) \end{cases}$$

where  $\Phi$  is defined by (47). Notice that the above controller is implicitly defined ( $\dot{p}$  is defined in terms of  $u$  and  $u$  in terms of  $\dot{p}$ ), we have to solve  $u$  by the following relation:

$$u = -\frac{1}{4}g_2^T(x, p)\frac{\partial V^T}{\partial x}(x) + v(x, p, \mu(p, \Phi(x, w, u, p))). \quad (58)$$

If there is a solution  $\hat{u}$  to (58), then the adaptive  $\mathcal{H}_\infty$ -control problem can be solved as state as follows:

**Proposition 5.8** *Consider the parameter-dependent system (26). Suppose there exists a non-negative function  $V : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  satisfying (32) for each  $\theta \in \Theta$ ; in addition, there exists continuous functions  $v(x, p, \delta)$  and  $\hat{u}(x, w, p)$  satisfying the matching conditions (57) for  $\delta \in T_\Theta(p)$  and (58), respectively. Then the adaptive  $\mathcal{H}_\infty$ -control problem has a solution; and such an adaptive control law is given by*

$$\begin{cases} \dot{p} = \mu(p, \Phi(x, w, u, p)) \\ u = \hat{u}(x, w, p) \end{cases} \quad (59)$$

It is remarked that for system (26), if  $g_2(x, \theta)$  is independent of  $\theta$ , i.e.,  $g_{2i}(x) = 0$  for  $i = 1, \dots, r$ , then  $\Phi(x, u, w, p)$  defined by (47) is independent of  $u$ ; thus the matching condition (58) is trivially satisfied. However, in general, the equation (58) could be hard to solve to get a meaningful  $u$  as the projection is particularly involved. Similar results to the above proposition and problems exist with the use of direct parameter projection. In the following, we will consider to get rid of the condition (58) by the use of over-parameterization technique in [29, 17].

In fact, the family of system (52) parameterized by  $\theta$  is a subclass of the following family of systems parameterized by two parameter vectors  $\theta$  and  $\vartheta$ :

$$G(\theta, \vartheta) : \begin{cases} \dot{x} = f(x, \theta) + g_1(x, \theta)w + g_2(x, \theta)u^*(x, p) + g_2(x, \vartheta)v \\ z = h(x) + k_1(x)w + k_2(x)(u^*(x, p) + v) \end{cases}$$

where  $(\theta, \vartheta) \in \Theta \times \Theta$ , and the updated law of  $p \in \Theta$  is to be decided. Notice that though  $\theta$  and  $\vartheta$  are treated as two unknown parameters in  $\Theta$ , but their true values are the same. Now define a positive definite function  $W : \mathbf{R}^n \times \Theta \times \Theta \rightarrow \mathbf{R}^+$  as:

$$W(x, p, q) = V(x, p) + \alpha((p - \theta)^T(p - \theta)^T + (q - \vartheta)^T(q - \vartheta)),$$

where the positive number  $\alpha$  is chosen such that

$$\max_{p, q \in \Theta; \theta, \vartheta \in \Theta} \{\alpha((p - \theta)^T(p - \theta)^T + (q - \vartheta)^T(q - \vartheta))\} < \epsilon.$$

for the given  $\epsilon > 0$ . Notice that  $\theta = \vartheta$  in the above formula. If  $p, q \in \Theta$ , then

$$\dot{W}(x, p, q) = \frac{\partial V}{\partial x}(x, p)(f(x, \theta) + g_1(x, \theta)w + g_2(x, \theta)u^*(x, p) + g_2(x, \vartheta)v) +$$



$$\begin{aligned}
& + \frac{\partial V}{\partial p}(x, p)\dot{p} + 2\alpha(p - \theta)^T \dot{p} + 2\alpha(q - \theta)^T \dot{q} \\
& \leq \|w\|^2 - \|z\|^2 + 2\alpha(p - \theta)^T (\dot{p} - \Phi_p(x, w, u^*(x, p), p)) + 2\alpha(q - \theta)^T (\dot{q} - \Phi_q(x, v, p)) + \\
& + \frac{\partial V}{\partial x}(x, p)g_2(x, q)v + \frac{\partial V}{\partial p}(x, p)\dot{p}
\end{aligned}$$

where  $\Phi_p : \mathbf{R}^n \times \mathbf{R}^{p_1} \times \mathbf{R}^{p_2} \times \Theta \rightarrow \mathbf{R}^r$  and  $\Phi_q : \mathbf{R}^n \times \mathbf{R}^{p_2} \times \Theta \rightarrow \mathbf{R}^r$  is defined as

$$\Phi_p(x, w, u, p) = \frac{1}{2\alpha} \begin{bmatrix} \frac{\partial V}{\partial x}(x, p)(f_1(x) + g_{11}(x)w + g_{21}(x)u) \\ \frac{\partial V}{\partial x}(x, p)(f_2(x) + g_{12}(x)w + g_{22}(x)u) \\ \vdots \\ \frac{\partial V}{\partial x}(x, p)(f_r(x) + g_{1r}(x)w + g_{2r}(x)u) \end{bmatrix} \quad (60)$$

and

$$\Phi_q(x, v, p) = \frac{1}{2\alpha} \begin{bmatrix} \frac{\partial V}{\partial x}(x, p)g_{21}(x)v \\ \frac{\partial V}{\partial x}(x, p)g_{22}(x)v \\ \vdots \\ \frac{\partial V}{\partial x}(x, p)g_{2r}(x)v \end{bmatrix}. \quad (61)$$

Therefore,

$$\dot{W}(x, p, q) \leq \|w\|^2 - \|z\|^2 \quad (62)$$

if the following conditions are satisfied:

$$\frac{\partial V}{\partial x}(x, p)g_2(x, q)v + \frac{\partial V}{\partial p}(x, p)\dot{p} \leq 0; \quad (63)$$

$$p, q \in \Theta; \quad (64)$$

$$(p - \theta)^T (\dot{p} - \Phi_p(x, w, u^*(x, p), p)) \leq 0; \quad (65)$$

$$(q - \theta)^T (\dot{q} - \Phi_q(x, v, p)) \leq 0. \quad (66)$$

The condition (63) is the matching condition. The other conditions can be guaranteed by the use of projection to choose the update law and the control function. From Theorem 3.3, we have that if the parameter update law is defined as

$$\begin{cases} \dot{p} = \mu(p, \Phi_p(x, w, u^*(x, p), p)) \\ \dot{q} = \mu(q, \Phi_q(x, v, p)) \end{cases}$$

where  $\mu$  is the vector projection with respect to the set  $\Theta$ , then the above conditions (64), (65), and (66) are satisfied.

Therefore, if there exists a function  $v(x, p, q, \delta)$  that satisfies the following inequality:

$$\frac{\partial V}{\partial x}(x, p)g_2(x, q)v + \frac{\partial V}{\partial p}(x, p)\delta \leq 0; \quad (67)$$

for  $\delta \in T_\Theta(p)$ , then (63) is satisfied with  $v(x, p, q, \dot{p})$  as  $\dot{p} \in T_\Theta(p)$  (see Corollary 3.8). Thus (62) is satisfied by the following adaptive law:

$$\begin{cases} \dot{p} = \mu(p, \Phi_p(x, w, u^*(x, p), p)) \\ \dot{q} = \mu(q, \Phi_q(x, v, p)) \\ u = -\frac{1}{4}g_2^T(x, p)\frac{\partial V^T}{\partial x}(x) + v(x, p, q, \dot{p}) \end{cases} \quad (68)$$

where  $\Phi_p$  and  $\Phi_q$  are defined by (60) and (61), respectively. We have the following Theorem.

**Theorem 5.9** *Consider the parameter-dependent system (26). Suppose there exists a non-negative function  $V : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  such that for each  $\theta \in \Theta$ ,*

$$\frac{\partial V}{\partial x}(x, \theta)f(x, \theta) + \frac{1}{4}\frac{\partial V}{\partial x}(x, \theta)(g_1(x, \theta)g_1^T(x, \theta) - g_2(x, \theta)g_2^T(x, \theta))\frac{\partial V^T}{\partial x}(x, \theta) + h^T(x)h(x) \leq 0; \quad (69)$$

*in addition, there exists a function  $v(x, p, q, \delta)$  satisfying the matching condition (67) for  $\delta \in T_\Theta(p)$ . Then the adaptive  $\mathcal{H}_\infty$ -control problem has a solution with given  $\epsilon > 0$ ; and such an adaptive control law is given by (68).*

In the same way, the direct parameter projection method can be used to construct adaptive control law.

**Theorem 5.10** *Consider the parameter-dependent system (26). Suppose there exists a non-negative function  $V : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  such that for each  $\theta \in \Theta$ , (69) is satisfied; in addition, there exists a function  $v(x, p, q, \delta)$  satisfying the matching condition (67) for  $\delta \in T_\Theta(p)$ . Then the adaptive  $\mathcal{H}_\infty$ -control problem has a solution with given  $\epsilon > 0$ ; and such an adaptive control law is given by*

$$\begin{cases} \dot{\pi} = \Phi_p(x, w, u^*(x, p), p) \\ \dot{\sigma} = \Phi_q(x, v, p) \\ p = \Pi(\pi) \\ q = \Pi(\sigma) \\ u = -\frac{1}{4}g_2^T(x, p)\frac{\partial V^T}{\partial x}(x) + v(x, p, q, \dot{p}) \end{cases} \quad (70)$$

where  $\Phi_p$  and  $\Phi_q$  are defined by (60) and (61), respectively, and  $\Pi$  is the direct parameter projection with respect to  $\Theta$ .

It is remarked that if (48) is satisfied, both (57) and (67) are satisfied with  $v = 0$ . To conclude this subsection, we will examine when the matching condition is satisfied. In fact, given the parameter-dependent  $\mathbf{C}^1$  positive definite storage function  $V(x, p)$ , then there exist two matrix-valued functions  $P, Q : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^{n \times n}$  such that:

$$V(x, p) = x^T Q(x, p)x, \quad \frac{\partial V}{\partial x}(x, p) = 2x^T P(x, p).$$

Thus, the matching condition (67) is satisfied if

$$2P(x, p)g_2(x, q)v = Q_d(x, p)\delta$$

where

$$Q_d(x, p) := - \begin{bmatrix} \frac{\partial Q}{\partial p_1}(x, p)x & \frac{\partial Q}{\partial p_2}(x, p)x & \cdots & \frac{\partial Q}{\partial p_r}(x, p)x \end{bmatrix}.$$

Therefore, if

$$Q_d(x, p)T_K(p) \subset P(x, p)\text{IM}(g_2(x, q))$$

for all  $x \in \mathbf{R}^n$  and  $p, q \in \Theta$ , then there exists a function  $v(x, p, q, \delta)$  such that the matching condition (67) is satisfied. A sufficient (restrictive) condition is that  $\text{RANK}(g_2(x, q)) = \text{RANK}(P(x, q)) = n$ . Similar matching conditions are also useful in other adaptive control problems [29, 28, 17]; for more discussions on such matching conditions, the reader is referred to [29, 28, 17]. In general, the matching conditions could be restrictive. To get rid of the matching condition (67), one can follow the adaptive controlled Lyapunov function approach to design tuning functions by considering instead modified systems [17]; however, some other restrictions need to be imposed as the parameter sets in this paper are bounded.

#### 5.4 Stability and Asymptotic Property of Adaptive $\mathcal{H}_\infty$ -Control Systems

In this subsection, we will examine stability and asymptotic properties of the adaptive  $\mathcal{H}_\infty$ -control systems. We will show that the adaptive  $\mathcal{H}_\infty$ -control systems with control laws constructed by the use of projections have some nice stability and asymptotic property once the disturbances are shut off.

A system is **zero-detectable** if the output  $z(t) = 0$ , then its state  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  for each  $\theta \in \Theta$ . Notice that the zero-detectability is equivalent to the detectability in linear case. In the following, we consider the asymptotic property of the adaptive control system without the disturbances. We take the adaptive control system with control law (44) as an example. The other case can be discussed similarly. We have the following theorem about the asymptotic stability and parameter convergence.

**Theorem 5.11** *Suppose the system (26) is zero-detectable for each  $\theta \in \Theta$ . Consider the adaptive  $\mathcal{H}_\infty$ -control systems with adaptive  $\mathcal{H}_\infty$ -controllers in terms of vector projection (i.e., (44), (49), and (68)). Suppose each storage function  $V$  satisfying the corresponding Hamilton-Jacobi inequality is positive definite for all  $\theta \in \Theta$ . If  $w = 0$ , then the adaptive  $\mathcal{H}_\infty$ -control system is Lyapunov stable; in addition, the states of the system satisfy that  $x(t) \rightarrow 0$  and  $\dot{p}(t) \rightarrow 0$  as  $t \rightarrow \infty$  for initial states  $x(0) \in \mathbf{R}^n$  and  $p(0) \in \Theta$ .*

**Proof** We just consider the case with adaptive control law (44). The other case can be considered similarly. Consider the closed loop system (26)-(44). Let  $w = 0$ ; take  $W : \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  defined in (39) as a Lyapunov function. Then (42) is satisfied. From (42), we have

$$\dot{W}(x, p) \leq -\|z\|^2 \leq 0.$$

Then the closed loop system is Lyapunov stable. Now if  $\dot{W}(x, p) = 0$ , then  $z(t) = 0$ , which implies  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  by detectability assumption. So LaSalle's theorem implies that the system satisfies  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  (see [33]).

On the other hand, as  $V(x)$  is a  $\mathbf{C}^1$  positive definite function, then

$$\lim_{t \rightarrow \infty} \frac{\partial V}{\partial x}(x(t)) = \frac{\partial V}{\partial x}(0) = 0,$$

then with the function  $\Phi$  defined by (41), one has

$$\lim_{t \rightarrow \infty} \Phi(x(t), 0, -\frac{1}{4}g_2^T(x(t), p) \frac{\partial V}{\partial x}(x(t))) = 0.$$

Therefore, by Theorem 3.3(ii),

$$\begin{aligned} 0 &\leq \lim_{t \rightarrow \infty} \left\| \mu(p(t), \Phi(x(t), 0, -\frac{1}{4}g_2^T(x(t), p(t))\frac{\partial V^T}{\partial x}(x(t))) \right\| \\ &\leq \lim_{t \rightarrow \infty} \left\| \Phi(x(t), 0, -\frac{1}{4}g_2^T(x(t), p)\frac{\partial V^T}{\partial x}(x(t))) \right\| = 0 \end{aligned}$$

From the update law (44), one has

$$\lim_{t \rightarrow \infty} \dot{p}(t) = \lim_{t \rightarrow \infty} \mu(p(t), (\Phi(x(t), 0, -\frac{1}{4}g_2^T(x(t), p(t))\frac{\partial V^T}{\partial x}(x(t)))) = 0$$

□

It is remarked that  $\dot{p}(t) \rightarrow 0$  does not implies  $p(t) \rightarrow p^*$  for some  $p^* \in \Theta$ . However, it is true under some stronger detectability assumption. This issue is out of scope of this paper.

## 6 Adaptive Robust Control of Uncertain Nonlinear Systems

It is generally recognized that a specific parametric model does not exactly describe a physical system primarily due to the neglected dynamics which are not captured by the uncertain real parameter. To handle this situation, one need to use adaptive robust control schemes which combine both robust control and adaptive control schemes. The area of adaptive robust control has been very active, see [12, 35, 23, 15, 16, 22, 11] and references therein for different treatments of this issue with various descriptions of the uncertainties.

In this section, we will consider the control problem of systems under both unknown constant parameter and  $\mathcal{L}_2$ -bounded uncertainty, which could be time-varying and dynamic. We will take advantage of the new development in robust control as well as some standard techniques in adaptive control. The uncertainty description is fairly general and standard in the robust control literature. Though in [16], similar uncertainty description is considered, the approach taken there is based on parameter-estimation, i.e., it is two-step design: (i) design the controller law with perfect knowledge of the parameter; (ii) design the robust adaptive estimator for the parameter, and the controller is implemented by the use of the estimated parameter. The BIBO property is proved for this type of adaptive scheme, and the achievable performance can be estimated *a posteriori*. In this section, we consider a different approach using Lyapunov design technique, and asymptotic stability is guaranteed for the resulting adaptive robust control system. The uncertain system is described as follows:

$$G(\theta) : \begin{cases} \dot{x} &= f(x, \theta) + g_1(x)w + g_2(x, \theta)u \\ z &= h(x) \\ w &= \Delta z \\ y &= x \end{cases} \quad (71)$$

where  $\theta$  is a  $r$ -dimensional vector of unknown constant parameters with  $\theta \in \Theta \in \mathbb{R}^r$ .  $\Delta$  is an unknown system which enters the system in a feedback fashion and is an  $\mathcal{L}_2$ -gain bounded. In

the following adaptive control problem, we will consider the parameter-dependence in the following fashion:

$$f(x, \theta) = f_0(x) + \sum_{i=1}^r \theta_i f_i(x)$$

$$g_2(x, \theta) = g_{20}(x) + \sum_{i=1}^r \theta_i g_{2i}(x).$$

Notice that we assume the multiplier  $g_1(x)$  of the uncertainty is independent of the unknown parameter. It is assumed that  $f_i, g_1, g_{2i}, h \in \mathbf{C}^0$ ,  $f_i(0) = 0, h(0) = 0$ ;  $x, u$ , and  $y$  are state, control input, and measured output, respectively. In this section, we consider the pure state-feedback solutions. In this section, we denote the admissible uncertainty set as  $\mathbf{B}\Delta$ , which is defined as follows.

**Definition 6.1** *For each  $\Delta \in \mathbf{B}\Delta$ , it is zero-detectable, and there defines a positive definite storage function  $U(\xi)$  continuously differentiable, where  $\xi \in \mathbf{R}^d$  is its state, such that for  $w = \Delta z$ , the following dissipation inequality is satisfied,*

$$\dot{U}(\xi) \leq \gamma^2 \|z\|^2 - \|w\|^2.$$

for some  $\gamma \in (0, 1)$ .

Therefore, for each  $\Delta \in \mathbf{B}\Delta$ , it has  $\mathcal{L}_2$ -gain  $< 1$ ; moreover, it is asymptotically stable with input  $z = 0$ . This assumption is standard in the robust control literature. For linear systems with bounded gains, the corresponding storage functions are always differentiable. The control problem is stated next.

**Definition 6.2 (Adaptive Robust Stabilization)** *Design an adaptive controller*

$$K : \begin{cases} \dot{p} = \phi(p, y, u) \\ u = \kappa(p)y \end{cases} \quad (72)$$

*such that the resulting closed-loop system with uncertainty  $\Delta \in \mathbf{B}\Delta$  is Lyapunov stable for all  $\theta \in \Theta$ . Furthermore, the states of the plant and uncertainty satisfy  $x(t) \rightarrow 0$  and  $\xi(t) \rightarrow 0$  with initial states  $x(0) \in \mathbf{R}^n$ ,  $\xi(0) \in \mathbf{R}^d$ , and  $p(0) \in \Theta$  as  $t \rightarrow \infty$  for all  $\Delta \in \mathbf{B}\Delta$  and all  $\theta \in \Theta$ .*

The block diagram of an adaptive robust control system is illustrated in Figure 3.

As the uncertainty  $\Delta$  is of  $\mathcal{L}_2$ -gain strictly bounded by 1, i.e., with  $w = \Delta z$ ,

$$\sup_{w \in \mathcal{L}_2[0, \infty), \|w\|_2 \neq 0} \frac{\|w\|_2}{\|z\|_2} < 1.$$

Then the small-gain theorem can be used to get the condition of robust I/O stability. However, the stabilization here is internal and asymptotic. There are more needed to guarantee the asymptotic stability. Nevertheless, the small-gain argument provides some clue to construct the Lyapunov function.

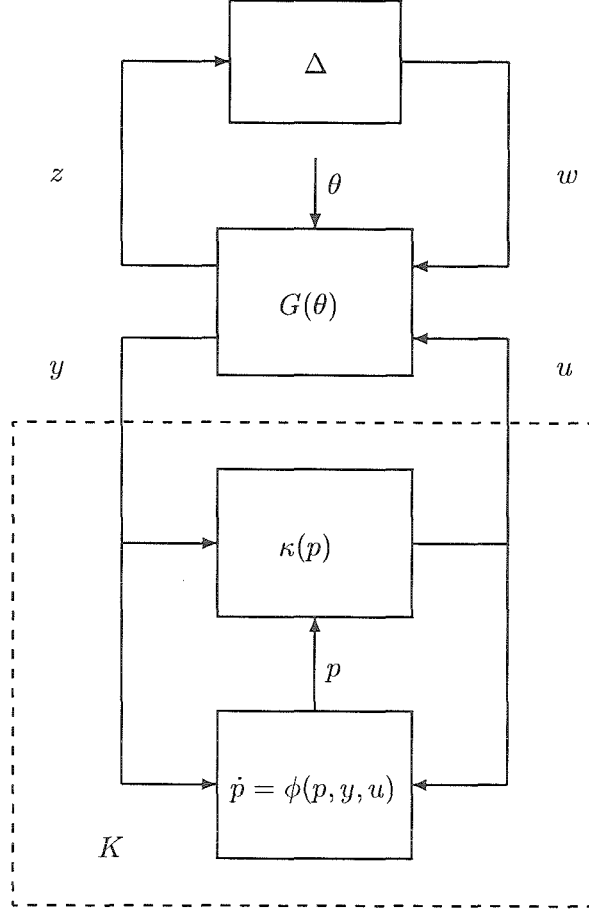


Figure 3: Adaptive Robust Control System

We first assume that there exists a positive definite function  $V : \mathbf{R}^n \rightarrow \mathbf{R}^+$  which is independent of  $\theta$  such that it satisfies the following Hamilton-Jacobi inequality<sup>3</sup>:

$$\frac{\partial V}{\partial x}(x)f(x, \theta) + \frac{1}{4} \frac{\partial V}{\partial x}(x)(g_1(x)g_1^T(x) - g_2(x, \theta)g_2^T(x, \theta)) \frac{\partial V^T}{\partial x}(x) + h^T(x)h(x) \leq 0. \quad (73)$$

for all  $x \in \mathbf{R}^n$ , which guarantees that the resulting parameter-dependent state-feedback system has  $\mathcal{L}_2$ -gain  $\leq 1$ .

Now take  $\Delta \in \mathbf{B}\Delta$ , then there exist a positive number  $\gamma < 1$  and a positive definite function  $U : \mathbf{R}^d \rightarrow \mathbf{R}^+$  such that

$$\dot{U}(\xi) \leq \gamma^2 \|z\|^2 - \|w\|^2$$

with  $w = \Delta z$ .

Motivated by the treatment of adaptive  $\mathcal{H}_\infty$ -control design, let  $W : \mathbf{R}^d \times \mathbf{R}^n \times \Theta \rightarrow \mathbf{R}^+$  be a positive definite function defined as

$$W(\xi, x, p) = \frac{2}{1 + \gamma^2} U(\xi) + V(x) + (p - \theta)^T (p - \theta).$$

<sup>3</sup>The case when  $V$  is dependent on the parameter can be considered similarly to the adaptive  $\mathcal{H}_\infty$ -control case.

Take  $W$  as a Lyapunov function candidate of the adaptive robust control system. Then

$$\begin{aligned}\dot{W}(\xi, x, p) &= \frac{2}{1+\gamma^2} \dot{U}(\xi) + \dot{V}(x) + 2(p-\theta)^T \dot{p} \\ &\leq \frac{2}{1+\gamma^2} (\gamma^2 \|z\|^2 - \|w\|^2) + \frac{\partial V}{\partial x}(x) (f(x, p) + g_1(x)w + g_2(x, p)u) + \\ &\quad + \sum_{i=1}^r \left\{ \frac{\partial V}{\partial x}(x) (\theta_i - p_i) (f_i(x) + g_{2i}(x)u) \right\} + 2(p-\theta)^T \dot{p}\end{aligned}$$

Notice that if  $p \in \Theta$ , by replacing the HJI (73) into the right hand side and using the completion of square, one has

$$\begin{aligned}\dot{W}(\xi, x, p) &\leq \frac{2}{1+\gamma^2} (\gamma^2 \|z\|^2 - \|w\|^2) + \|w\|^2 - \|z\|^2 + \frac{\partial V}{\partial x}(x) g_2(x, p) (u(t) + \frac{1}{4} g_2^T(x, p) \frac{\partial V^T}{\partial x}(x)) - \\ &\quad + \left\| w(t) - \frac{1}{2} g_1^T(x) \frac{\partial V^T}{\partial x}(x) \right\|^2 + 2(p-\theta)^T (\dot{p} - \Phi(x, u)),\end{aligned}\tag{74}$$

where  $\Phi : \mathbf{R}^n \times \mathbf{R}^{p_1} \times \mathbf{R}^{p_2} \rightarrow \mathbf{R}^r$  is defined as

$$\Phi(x, u) = \frac{1}{2} \begin{bmatrix} \frac{\partial V}{\partial x}(x) (f_1(x) + g_{21}(x)u) \\ \frac{\partial V}{\partial x}(x) (f_2(x) + g_{22}(x)u) \\ \vdots \\ \frac{\partial V}{\partial x}(x) (f_r(x) + g_{2r}(x)u) \end{bmatrix}.\tag{75}$$

From (74), we have the following conclusion:

$$\dot{W}(x, p) \leq -\frac{1-\gamma^2}{1+\gamma^2} (\|z\|^2 + \|w\|^2) \leq 0,\tag{76}$$

if the following conditions are satisfied:

$$p(t) \in \Theta, \forall t \in \mathbf{R}^+;\tag{77}$$

$$u = -\frac{1}{4} g_2^T(x, p) \frac{\partial V^T}{\partial x}(x);\tag{78}$$

$$(p-\theta)^T (\dot{p} - \Phi(x, u)) \leq 0.\tag{79}$$

From Theorem 3.3, the above conditions are satisfied by choosing the update law and the control function by the use of vector projection. Therefore, (76) is satisfied by the following adaptive law:

$$\begin{cases} \dot{p} = \mu(p, \Phi(x, -\frac{1}{2} g_2^T(x, p) \frac{\partial V^T}{\partial x}(x))) \\ u = -\frac{1}{2} g_2^T(x, p) \frac{\partial V^T}{\partial x}(x) \end{cases}\tag{80}$$

where  $\Phi$  is defined by (75).

Next, we analyze the stability of the adaptive robust control system (71)-(80). Take the positive definite function  $W$  as the Lyapunov function. Then it satisfies (76). Therefore, the system (71)-(80) is Lyapunov stable. Now  $\dot{W}(x, p) = 0$  implies  $z = 0$  and  $w = 0$ . If we further assume that the plant is zero-detectable for all  $\theta \in \Theta$ , then together with the detectability assumption on uncertainty  $\Delta$ , we have that  $x(t) \rightarrow 0$  and  $\xi(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Therefore, from LaSalle's theorem (see, e.g., [33]), we have the asymptotic stability assertion. The above discussion is summarized in the following theorem.

**Theorem 6.3** *Consider the parameter-dependent uncertain system (71); suppose the plant is zero-detectable and the uncertainty  $\Delta \in \mathbf{B}\Delta$ . In addition, suppose there exists a positive definite function  $V : \mathbf{R}^n \rightarrow \mathbf{R}^+$  such that for each  $\theta \in \Theta$ , (73) is satisfied. Then the adaptive robust stabilization problem has a solution. And such an adaptive control law is given by (80).*

Use the same argument as in Section 5.4, one has the following parameter convergence property.

**Corollary 6.4** *Under the adaptive control law is given by (80), the adaptive robust control system has the following property:*

$$\lim_{t \rightarrow \infty} \dot{p}(t) = 0.$$

It is remarked that the vector projection technique used here makes it easy to show the stability using Lyapunov technique. The direct parameter projection technique does not provide such convenience because of the integral relation provided in Theorem 3.7.

## 7 An Illustrative Example and Some Remarks

### 7.1 A Numerical Example

Consider the following second order system:

$$\begin{cases} \dot{x}_1 = x_2 + \frac{1}{\sqrt{5}}w_1 \\ \dot{x}_2 = -(1 + \theta_1)x_1 - (2 + \theta_1)x_2 + \frac{1}{\sqrt{5}}w_2 + (-1 + \theta_2)u \\ z = \begin{bmatrix} x \\ u \end{bmatrix} \end{cases}$$

where  $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbf{R}^2$  and  $w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbf{R}^2$  are the state and disturbance vectors, and the parameter vector:

$$\theta := (\theta_1, \theta_2) \in \Theta := [-0.04, 5] \times [-5, 0.04].$$

So parameter set  $\Theta$  is convex and compact. We need to design a adaptive controller such that the resulting system satisfied the adaptive  $\mathcal{H}_\infty$  performance (28).

First the Hamilton-Jacobi inequality is

$$\frac{\partial V}{\partial x}(x, \theta) \begin{bmatrix} x_2 \\ -(1 + \theta_1)x_1 - (2 + \theta_1)x_2 \end{bmatrix} + \frac{1}{4} \frac{\partial V}{\partial x}(x, \theta) \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 - (1 - \theta_2)^2 \end{bmatrix} \frac{\partial V^T}{\partial x}(x, \theta) + x^T x \leq 0$$



for  $\theta \in \Theta$ . The above inequality has a positive definite solution:

$$V(x) = x^T P x, \quad P = \begin{bmatrix} 3.95 & 1.46 \\ 1.46 & 1.01 \end{bmatrix}.$$

Notice that the storage function  $V$  is independent of parameter  $\theta$ . Then the results in Section 4.2 can be used. First, a gain-scheduled  $\mathcal{H}_\infty$ -controller is given by

$$u = -\frac{1}{2} \begin{bmatrix} 0 & \theta_2 - 1 \end{bmatrix} \frac{\partial V^T}{\partial x}(x) = (1 - \theta_2)(1.46x_1 + 1.01x_2). \quad (81)$$

Take  $\epsilon = 0.15$  in (28), then  $\alpha$  in (41) can be taken as 0.02. The function  $\Phi(x, u)$  defined by (41) is as follows:

$$\Phi(x, u) = 50 \begin{bmatrix} -(1.46x_1 + 1.01x_2)(x_1 + x_2) \\ (1.46x_1 + 1.01x_2)u \end{bmatrix},$$

where  $\Phi$  is independent of disturbance because  $g_1(x, \theta)$  in the plant is independent of  $\theta$ . Therefore, we can get a state-feedback adaptive control solution. In fact, using Theorems 5.4 and 5.5, one has the following two adaptive controllers with respect to vector and direct parameter projection techniques, respectively:

$$\begin{cases} \dot{p} = \mu_\Theta(p, \Phi(x, u)) \\ u = (1 - p_2)(1.46x_1 + 1.01x_2) \end{cases} \quad (82)$$

and

$$\begin{cases} \dot{\pi} = \Phi(x, u) \\ p = \Pi_\Theta(\pi) \\ u = (1 - p_2)(1.46x_1 + 1.01x_2) \end{cases} \quad (83)$$

with  $p = (p_1, p_2) \in \mathbf{R}^2$ . Suppose the true parameter is  $\theta = (2.5, -2.5)$ . If the parameter is known, the deterministic state-feedback controller (81) is used, and the simulation results for the deterministic  $\mathcal{H}_\infty$ -control system are in Fig. 4. If the parameter is unknown, the adaptive control schemes (82) and (83) are used; the initial guess for the parameter is  $p(0) = (3.3, -4.0)$ . The simulation results for the two adaptive  $\mathcal{H}_\infty$ -control systems are as in Figs. 5 and 6, respectively. In those simulations, the disturbances  $w_1$  is sine wave (frequency = 3.73, peak = 2.00) and  $w_2$  is sawtooth wave (frequency = 7.77, peak = 3.00). The simulation results for both adaptive controllers are almost identical. But the state response of the deterministic control  $\mathcal{H}_\infty$ -system is different from either of the adaptive  $\mathcal{H}_\infty$ -control systems as expected.

## 7.2 Some Remarks

In this paper, we considered the adaptive  $\mathcal{H}_\infty$ -control and adaptive robust control problems using dissipation theoretical methods; the sufficient conditions are characterized as the non-negative solutions of parameter-dependent Hamilton-Jacobi inequalities; the storage functions are explicitly constructed to generate the adaptive  $\mathcal{H}_\infty$ -control laws. The projection techniques play an important role in derivation of the adaptive controllers.

As mentioned in Section 4.1, as far as dissipativity is concerned, the adaptive  $\mathcal{H}_\infty$ -control problem considered in this paper is closely related to the minimax adaptive control problem by

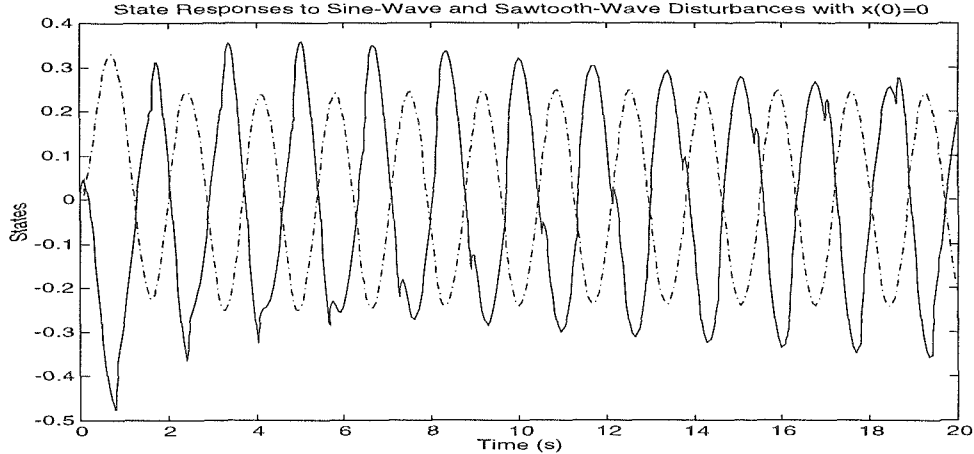


Figure 4: Deterministic  $\mathcal{H}_\infty$ -Controller

Didinsky-Basar [7]. In [7], the authors use the cost-to-go function method from dynamical game theory to derive the adaptive controllers; the emphasis is on the optimality of the minimax adaptive control problem; the optimal solutions are characterized in [7] and have some nice properties [5]. It is known that the optimal solution to the minimax problem is also a solution to the adaptive  $\mathcal{H}_\infty$ -control problem, but the converse is not necessarily true. This provides some flexibility to the adaptive  $\mathcal{H}_\infty$ -control design. In this paper, we employ this flexibility to derive the adaptive  $\mathcal{H}_\infty$ -control laws by choosing simpler storage functions; in addition to the simplicity of adaptive control law derivation, there are several other features for the results in this paper, which are compared with the results in [7] as follows:

- **Assumption on Plants:** A useful assumption on the parameter-dependent plant (26) for the full information feedback solutions in [7] is  $g_1(x, \theta) = I$ , but there is no need for such assumption in this paper.
- **Complexity of Controllers:** The orders of dynamics of the adaptive control laws in this paper are as low as the dimensions of parameter-vectors, they are much lower than those in [7]. For example, if the dimension of the parameter vector is  $r$ ; with the same condition, e.g., the solvability of HJI (38), the dynamics order of either of the controllers provided by Theorems 5.4 and 5.5 in this paper is  $r$ , while the controllers provided in [7] have orders at least  $r + r^2$ .
- **Information Patterns:** If  $g_1(x, \theta)$  is independent of the parameter  $\theta$ , then the results in this paper give pure state feedback solution. Therefore, with the assumption  $g_1(x, \theta) = I$  used in [7] on the plant (26), the controllers (with order  $r$ ) derived in this paper only use state information; however, the general controllers derived in [7] use both state and disturbance information. Though in [7], the solutions can be approximated with controllers using pure state information, but the dimension of the controllers increases to  $n + r + r^2$ .

It should be emphasized that all the conditions for solvability are sufficient in this paper; in particular, in the case when the storage functions are dependent on the parameters, the required matching condition (57) could be restrictive. The existence and uniqueness of solutions for adaptive

$\mathcal{H}_\infty$ -control and adaptive robust control systems is not discussed explicitly here. This issue is out of the scope of this paper; the reader is also referred to [27] for some related discussions. Another issue which is not considered in this paper is the computational implication of the parameter-dependent Hamilton-Jacobi inequalities (42), (38), and (73); like deterministic nonlinear  $\mathcal{H}_\infty$ -control results [36, 3, 13, 18], additional efforts are needed to find efficient computational algorithms; for low dimensional problems, the computation can be conducted by the use of parameter space gridding technique and finite difference schemes [14].

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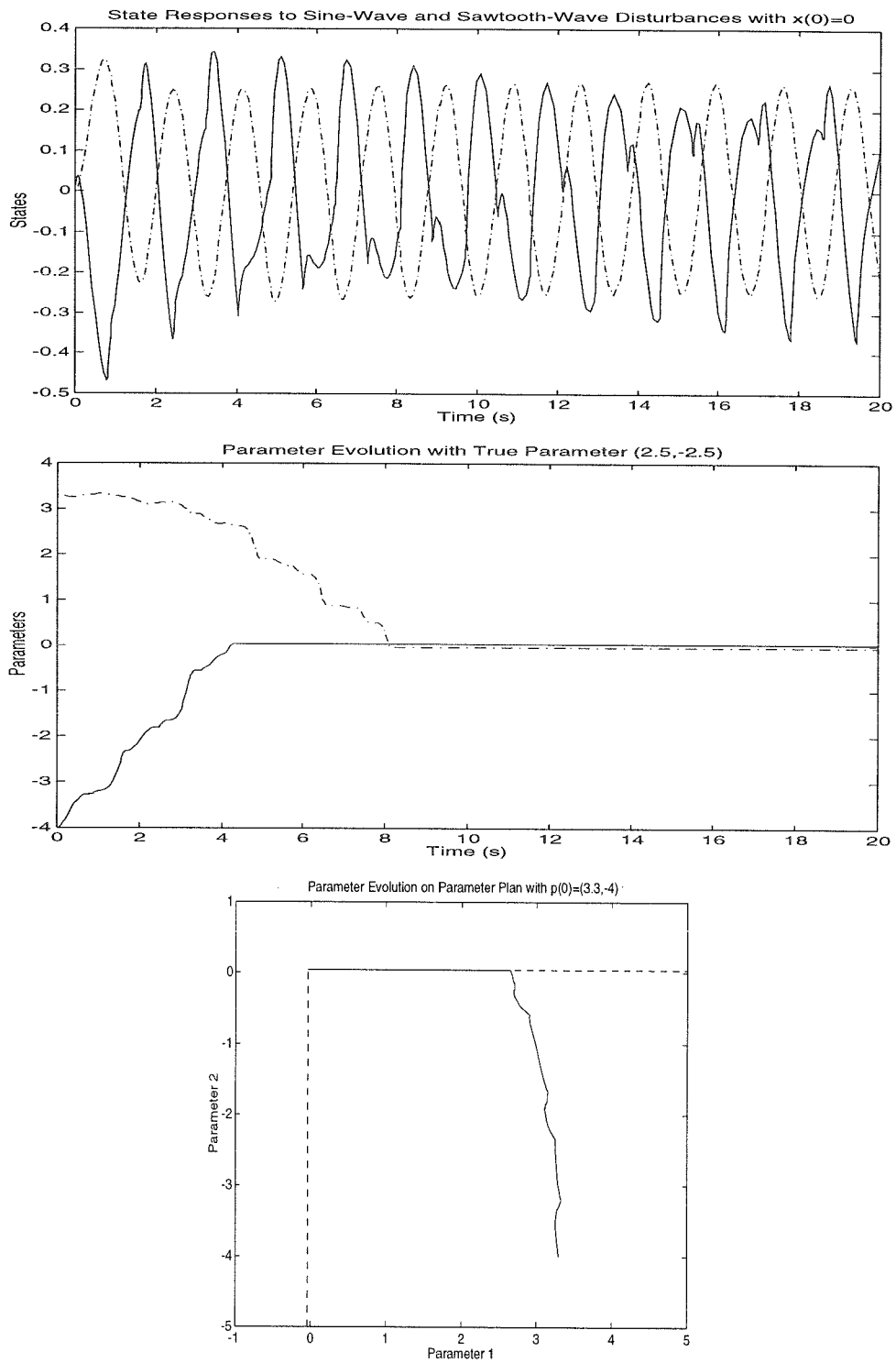


Figure 5: Adaptive  $\mathcal{H}_\infty$ -Controller with Vector Projection

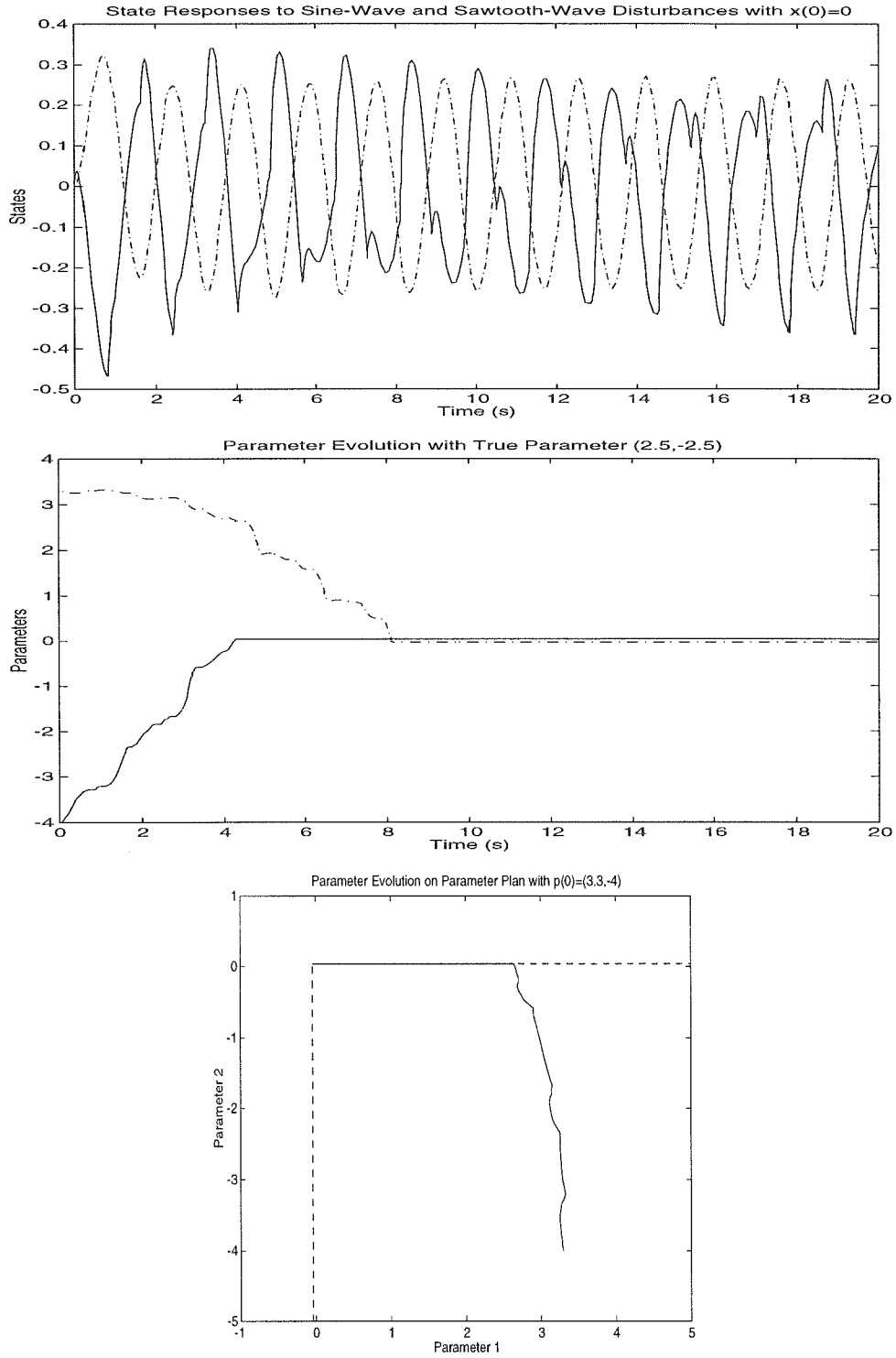


Figure 6: Adaptive  $\mathcal{H}_\infty$ -Controller with Direct Parameter Projection